Chapter 5

Wavelets with compact support

1. Introduction

In this chapter we study wavelets with compact support. It is quite easy to see that if the father wavelet or scaling function φ has compact support, then the filter α has compact support as well, i.e., it is a finite sequence. At least in the case where φ decays sufficiently rapidly at $\pm\infty$ the converse also holds.

First we consider some results that are somewhat more general than what we actually need for the analysis of wavelets.

2. Dilation equations

In Chapter 4 we found that a crucial property of the father wavelet or scaling function φ determining a multiresolution is that it satisfies the dilation equation

$$\varphi = 2\sum_{k\in\mathbb{Z}} \alpha(k)\varphi(2\bullet - k). \tag{5.1}$$

In this section we look at some properties of a generalization of this equation.

First we observe that every sequence $(\alpha(k))_{k\in\mathbb{Z}}$ can be identified with a Radon measure, i.e. a local measure, by defining $\alpha_\diamond(E) = \sum_{k\in E}\alpha(k)$ for every bounded set $E \in \mathbb{R}$. If we assume that $\alpha \in l^1(\mathbb{Z})$, then $\alpha_\diamond$ is a finite measure. Now (5.1) can be rewritten as

$$\varphi = 2(\alpha_\diamond * \varphi)(2\bullet)$$

where $*$ denotes convolution. Thus the generalized equation that we will be looking at here is

$$f = \rho(\mu * f)(\rho\bullet), \tag{5.2}$$

where ρ is some real number > 1 and $\mu \in M(\mathbb{R};\mathbb{C})$, i.e., μ is a complex measure on $\mathbb{R}$.

First we prove an auxiliary result on the convergence of products. We use the notation $|\bullet|_+ = \max\{0, \bullet\}$.

Lemma 5.1. *Let $\mu \in M(\mathbb{R};\mathbb{C})$ be such that $\mu(\mathbb{R}) = 1$ and $\int_{\mathbb{R}}(|\log|x||_+ + 1)|\mu|(\mathrm{d}x) < \infty$. Then the product $\prod_{k=1}^{\infty} \hat{\mu}(\rho^{-k}\underline{\omega})$ converges uniformly on compact subsets of $\mathbb{R}$ towards a continuous function.*

Proof. Since $\mu(\mathbb{R}) = 1$ we have $\hat{\mu}(\underline{\xi}) - 1 = \int_{\mathbb{R}}(\mathrm{e}^{-\mathrm{i}2\pi x\underline{\xi}} - 1)\mu(\mathrm{d}x)$, and hence

$$|\hat{\mu}(\xi) - 1| \le \int_{\mathbb{R}} 2|\sin(\pi x\xi)||\mu|(\mathrm{d}x), \quad \xi \in \mathbb{R}.$$

Let m be a positive integer and let $\omega \in \mathbb{R}$. Now it is clear from the preceding inequality, Fubini's theorem, and the fact that $|\sin(\underline{t})| \le \min\{1, |\underline{t}|\}$ that

$$\begin{aligned}
\sum_{k=m}^{\infty}\left|\hat{\mu}(\rho^{-k}\omega) - 1\right| &\le 2\sum_{k=m}^{\infty}\int_{\mathbb{R}}|\sin(\rho^{-k}\pi x\omega)||\mu|(\mathrm{d}x) \\
&\le 2\int_{\mathbb{R}}\Biggl(\sum_{k=m}^{\lceil\log_\rho(\pi|x\omega|)\rceil} 1 + \sum_{k=\max\{\lceil\log_\rho(\pi|x\omega|)\rceil+1, m\}}^{\infty}\rho^{-k}\pi|x\omega|\Biggr)|\mu|(\mathrm{d}x) \\
&\le 2\int_{\mathbb{R}}\left(\left|\lceil\log_\rho(\pi|x\omega|)\rceil + 1 - m\right|_+ + \tfrac{1}{\rho-1}\rho^{-|m-\lceil\log_\rho(\pi|x\omega|)\rceil-1|_+}\right)|\mu|(\mathrm{d}x).
\end{aligned}$$

From this inequality we get the uniform convergence on compact intervals of the series and this implies the claim of the lemma by [**1**, Th. 15.4]. $\square$

We proceed with an easy result.

Proposition 5.2. *Assume $\rho > 1$ and that $\mu \in M(\mathbb{R};\mathbb{C})$ satisfies $|\mu(\mathbb{R})| \le 1$ and $\int_{\mathbb{R}}\bigl(|\log(|x||_+ + 1\bigr)|\mu|(\mathrm{d}x) < \infty$ if $|\mu(\mathbb{R})| = 1$. If equation (5.2) has a nontrivial solution $f \in L^1(\mathbb{R};\mathbb{C})$, then $\mu(\mathbb{R}) = 1$ and this solution is unique in $L^1(\mathbb{R};\mathbb{C})$ up to a multiplicative constant.*

Proof. Taking Fourier transforms of both sides of (5.2) we get

$$\hat{f}(\rho\underline{\omega}) = \hat{\mu}(\underline{\omega})\hat{f}(\underline{\omega}). \tag{5.3}$$

If $|\mu(\mathbb{R})| < 1$, then it is clear that $\lim_{m\to\infty}\prod_{j=1}^{m}|\hat{\mu}(2^{-j}\omega)| = 0$ for every $\omega \in \mathbb{R}$. Thus we see from (5.3) that $\hat{f}(\omega) = 0$ for all $\omega \in \mathbb{R}$, so we can have no nontrivial solution f.

Suppose next that $|\mu(\mathbb{R})| = 1$. If $\hat{f}(0) \neq 0$, then we conclude from (5.3) that $\mu(\mathbb{R}) = \hat{\mu}(0) = \hat{f}(0)/\hat{f}(0) = 1$. If $\hat{f}(0) = 0$ then we have

$$|\hat{f}(\omega)| = \lim_{m\to\infty} \prod_{k=1}^{m} |\hat{\mu}(\rho^{-k}\omega)||\hat{f}(\rho^{-m}\omega)| = 0, \quad \omega \in \mathbb{R}.$$

because the product $\prod_{k=1}^{\infty} |\hat{\mu}(\rho^{-k}\omega)|$ converges by Lemma 5.1. Thus we see that f is identically 0.

If now $\mu(\mathbb{R}) = 1$, then we have by lemma 5.1 and (5.3) that

$$\hat{f}(\omega) = \hat{f}(0) \prod_{k=1}^{\infty} \hat{\mu}(\rho^{-k}\omega), \quad \omega \in \mathbb{R},$$

and we see that f is unique up to the multiplicative constant $\hat{f}(0)$. □

Next we consider the case where μ in (5.2) has compact support. First we prove an auxiliary result on how the support of $(\mu * f)(\rho\bullet)$ is related to the supports of μ and f.

Lemma 5.3. *Assume that $\rho > 1$, $\mu \in M(\mathbb{R};\mathbb{C})$ with* $\operatorname{supp}(\mu) \subset [M_-, M_+]$ *and that $f \in L^1(\mathbb{R};\mathbb{C})$ with* $\operatorname{supp}(f) \subset [F_-, F_+]$. *Then*

$$\operatorname{supp}\Big((\mu * f)(\rho\bullet)\Big) \subset \Big[\tfrac{F_- + M_-}{\rho}, \tfrac{F_+ + M_+}{\rho}\Big]. \tag{5.4}$$

Proof. Let $x < (M_- + F_-)/\rho$. Then $\rho x - t < M_- + F_- - t \le F_-$ if $t \ge M_-$. Similarly when $x > (M_+ + F_+)/\rho$ we have $\rho x - t > M_+ + F_+ - t \ge F_+$ if $t \le M_+$. This gives the desired conclusion. □

Since the previous result says that the operator $f \to (\mu * f)(\rho\bullet)$ forces the support closer to that of μ it is natural to expect that if μ has compact support and there is a solution of (5.2), then this solution has compact support as well. This turns out to be the case, at least if f is integrable.

Proposition 5.4. *Assume that $\rho > 1$, $\mu \in M(\mathbb{R};\mathbb{C})$ has compact support contained in the interval $[M_-, M_+]$ and that $\mu(\mathbb{R}) = 1$. If $f \in L^1(\mathbb{R};\mathbb{C})$ satisfies* (5.2)*, then f has compact support contained in the interval $[\frac{M_-}{\rho-1}, \frac{M_+}{\rho-1}]$.*

Proof. Let $f \in L^1(\mathbb{R};\mathbb{C})$ be some nontrivial function that satisfies (5.2). If we can show that f has compact support, then it follows from repeated applications of Lemma 5.3 that the support is contained in the desired interval.

Let us for simplicity assume that $M_- < 0$ and that $M_+ > 0$ Let $m \ge 0$ be an integer and let $f_m = f - f\chi_{[\rho^m M_-, \rho^m M_+]}$. Moreover, we define the linear operator $T : L^1(\mathbb{R};\mathbb{C}) \to L^1(\mathbb{R};\mathbb{C})$ by $T(g) = \rho(\mu * g)(\rho\bullet)$. If we apply Lemma 5.3 m times we see that $T^m(f - f_m)$ has support contained in the interval $[\frac{\rho M_-}{\rho-1}, \frac{\rho M_+}{\rho-1}]$. On the other hand we have $T^m(f_m) = f - T^m(f - f_m)$, and this means that

$$f(x) = T^m(f_m)(x), \quad x \notin \Big[\frac{\rho M_-}{\rho - 1}, \frac{\rho M_+}{\rho - 1}\Big]. \tag{5.5}$$

Moreover, we easily see that

$$\widehat{T^m(f_m)}(\underline{\omega}) = \prod_{k=1}^{m} \hat{\mu}(\rho^{-k}\underline{\omega})\widehat{f_m}(\rho^{-m}\underline{\omega}). \tag{5.6}$$

Let h be some infinitely many times differentiable function with support contained in $[-1, 1]$ and let $h_\lambda = \lambda h(\lambda \bullet)$, $\lambda > 0$. Now it follows from the inversion theorem for Fourier transforms (Theorem 2.3.(b)), (5.5) and (5.6) that

(5.7)
$$\int_{\mathbb{R}} h_\lambda(x - t)T^m(f_m)(t)\,\mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi x\omega}\widehat{h_\lambda}(\omega) \prod_{k=1}^{m} \hat{\mu}(\rho^{-k}\omega)\widehat{f_m}(\rho^{-m}\omega)\,\mathrm{d}\omega.$$

Now we know by Lemma 5.1 that $|\prod_{k=1}^{\infty} \hat{\mu}(\rho^{-k}\omega)|$ is bounded when $|\omega| \leq 1$. Then it follows for all $m \geq 1$ and $\omega \in \mathbb{R}$ that

$$\Big|\prod_{k=1}^{m} \hat{\mu}(\rho^{-k}\omega)\Big| \leq (\sup_{\omega\in\mathbb{R}}|\hat{\mu}(\omega|)^{\lceil \log_2(|\omega|)\rceil} \sup_{|\xi|\leq 1}|\prod_{k=1}^{\infty} \hat{\mu}(\rho^{-k}\xi)| \leq C(|\omega| + 1)^C$$

where C is some constant. Since h is infinitely many times differentiable, it follows that

$$\int_R |\widehat{h_\lambda}(\omega)|(|\omega| + 1)^C \,\mathrm{d}\omega < \infty,$$

and therefore it follows from (5.7), the dominated convergence theorem and from the fact that $f_m \to 0$ in $L^1(\mathbb{R};\mathbb{C})$ and hence $\widehat{f_m} \to 0$ in $L^\infty(\mathbb{R};\mathbb{C})$ as $m \to \infty$ that

$$\lim_{m\to\infty} \int_{\mathbb{R}} h_\lambda(x - t)T^m(f_m)(t)\,\mathrm{d}t = 0, \quad x \in \mathbb{R}, \quad k \geq 1.$$

But when we let $\lambda \to \infty$ we see from (5.5) that f must have compact support. □

3. Construction of wavelets with compact support

When performing calculations with the filter α on some real data it is clearly advantageous to have the sequence α to be real. This requirement we will make throughout this section where we want to find suitable sequences α that generate multiresolutions. From Theorems 4.12 and 4.13 we see that α must satisfy (4.11), (4.46), and (4.48).

We get the following characterization of the Fourier transform of filters α with compact (i.e., finite) support.

Theorem 5.5. *Let $\{\alpha(k)\}_{k\in\mathbb{Z}}$ be a sequence of real numbers with only finitely many nonzero terms. Then (4.11) and (4.46) hold if and only if*

$$\hat{\alpha}(\underline{\omega}) = \left(\tfrac{1}{2}\big(1+\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}}\big)\right)^N Q(\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}})\mathrm{e}^{-\mathrm{i}2\pi L\underline{\omega}}, \tag{5.8}$$

where $N \geq 1$, $L \in \mathbb{Z}$ and Q is a polynomial with real coefficients such that

$$\begin{aligned}\big|Q(\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}})\big|^2 = \sum_{k=0}^{N-1}\binom{N+k-1}{k}\sin(\pi\underline{\omega})^{2k}&\\ + \sin(\pi\underline{\omega})^{2N}R\big(\cos(2\pi\underline{\omega})\big),&\end{aligned} \tag{5.9}$$

where R is an odd real polynomial.

Proof. Suppose first that (4.11) and (4.46) hold. Since we require that $\hat{\alpha}(0) = 1$, it follows from (4.11) that $\hat{\alpha}(\frac{1}{2}) = 0$. In order to see that $\hat{\alpha}$ can be written in the form (5.8) we argue as follows: For some integer L the function $\underline{z}^{-L}\sum_{k\in\mathbb{Z}}\alpha(k)\underline{z}^k$ is a polynomial and this polynomial vanishes in the point $z = -1$. Thus it can be written in the form $(\frac{1}{2}(1+\underline{z}))^N Q(\underline{z})$ where $N \geq 1$ and Q is a real polynomial. Substituting $\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}}$ for $\underline{z}$ we get (5.8).

If $Q(\underline{z}) = \sum_{j=0}^M q_j\underline{z}^j$, then we have

$$\begin{aligned}|Q(\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}})|^2 = \sum_{k=-M}^{M}\tilde{q}_k\mathrm{e}^{-\mathrm{i}2\pi k\underline{\omega}} = \tilde{q}_0 + \sum_{k=1}^{M}\tilde{q}_k(\mathrm{e}^{-\mathrm{i}2\pi k\underline{\omega}} + \mathrm{e}^{-\mathrm{i}2\pi k\underline{\omega}})&\\ = \tilde{q}_0 + 2\sum_{k=1}^{M}\tilde{q}_k\cos(2\pi k\underline{\omega}),&\end{aligned}$$

since $\tilde{q}_{-k} = \tilde{q}_k$ for all k because $\tilde{q}_k = \sum_{j=\max\{0,-k\}}^{\min\{M,M-k\}} q_j\overline{q_{j+k}}$ and the coefficients $\underline{q_j}$ in Q are real. Since every term $\cos(2\pi k\underline{\omega})$ can be written as a polynomial in $\cos(2\pi\underline{\omega})$ (use De Moivre's formula and $\sin(2\pi\underline{\omega})^2 = 1-\cos(2\pi\underline{\omega})^2$) or equivalently as a polynomial in $\sin(\pi\underline{\omega})^2$ we see that there exists a polynomial P such that

$$|Q(\mathrm{e}^{\mathrm{i}2\pi\underline{\omega}})|^2 = P(\sin(\pi\underline{\omega})^2). \tag{5.10}$$

Since $\sin(\pi(\underline{\omega}+\frac{1}{2}))^2 = \cos(\pi\underline{\omega})^2 = 1-\sin(\pi\underline{\omega})^2$ and $|\frac{1}{2}(1+\mathrm{e}^{\mathrm{i}2\pi\underline{\omega}})| = \cos(\pi\underline{\omega})^2$ it follows from (4.11) that

$$(1-\underline{z})^N P(\underline{z}) + \underline{z}^N P(1-\underline{z}) = 1, \tag{5.11}$$

on the interval $[0,1]$ and therefore also on $\mathbb{R}$. We can write P in the form $P(\underline{z}) = \sum_{j=0}^{N-1} p_j\underline{z}^j + \bullet^N R_0(\underline{z})$. Inserting this expression into (5.11) we get the following system of equations for the coefficients p_j,

$$p_0 = 1,$$
$$p_k = \sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{N}{k-j}p_j.$$

Next have to we check that the solution of this recursive system of equations is

$$p_k = \binom{N+k-1}{k}, \quad 0 \le k \le N-1.$$

For $k = 0$ this is certainly the case and an induction argument works because we have

$$\begin{aligned}
&\sum_{j=0}^{k-1}(-1)^{k-j+1}\binom{N}{k-j}\binom{N+j-1}{j} \\
&= \frac{N}{k!}(-1)^{k+1}\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j\frac{(N+j-1)!}{(N+j-k)!}x^{N+j-k}\Big|_{x=1} \\
&= \frac{N}{k!}(-1)^{k+1}\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j\tfrac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}x^{N+j-1}\Big|_{x=1} \\
&= \frac{N}{k!}(-1)^{k+1}\tfrac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left(x^{N-1}(1-x)^k - (-1)^k x^{N+k-1}\right)\Big|_{x=1} \\
&= \frac{N}{k!}\tfrac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}x^{N+k-1}\Big|_{x=1} = \binom{N+k-1}{k}.
\end{aligned}$$

Thus we define the polynomial P_N by

$$P_N(\underline{z}) = \sum_{k=0}^{N-1}\binom{N+k-1}{k}x^k. \tag{5.12}$$

Next we observe that this polynomial is in fact a solution of (5.11), because the construction of the coeffiecients p_k guarantees that

$$(1-\underline{z})^N P_N(\underline{z}) + \underline{z}^N P_N(1-\underline{z}) - 1 = \underline{z}^N V(1-\underline{z}), \tag{5.13}$$

where V is polynomial of at most degree $N-1$. But then it follows that

$$\underline{z}^N V(1-\underline{z}) = (1-\underline{z})^N V(\underline{z}). \tag{5.14}$$

It follows from a calculation similar to the one used for finding the coefficients p_k, that V is identically zero since it is of a most degree $N-1$.

The original polynomial P was written in the form $P = P_N(\underline{z}) + \underline{z}^N R_0(\underline{z})$. If we insert this expression in (5.11) we conclude that

$$(1-\underline{z})^N\underline{z}^N R_0(\underline{z}) + (1-\underline{z})^N\underline{z}^N R_0(1-\underline{z}) = 0, \tag{5.15}$$

that is $R_0(\underline{z}) = -R_0(1-\underline{z})$ and this implies that $R_0(\underline{z}) = R(1-2\underline{z})$ where R is an odd polynomial. But this is exactly what we wanted to prove.

The converse goes in exactly the same way. $\square$

If we want to construct a filter sequence α, one possibility is to use Theorem 5.5. But then we must be able to find the trigonometric polynomial $Q(\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}})$ if $|Q(\mathrm{e}^{-\mathrm{i}2\pi\omega})|^2$ is known. This classical result is given in the next lemma.

Lemma 5.6. *Assume that* $A(\underline{\omega}) = \sum_{k=-M}^{M} a_k \mathrm{e}^{-\mathrm{i}2\pi k\underline{\omega}}$, *where* $a_k = a_{-k} \in \mathbb{R}$ *for* $k = 0, 1, \ldots, M$, *is nonnegative and* $A_M \neq 0$. *Then the* $2M$ *zeros of the polynomial* $\sum_{k=-M}^{M} a_k \underline{\omega}^{k+M}$ *are of the form* w_j, $\overline{w_j}$, w_j^{-1}, $\overline{w_j}^{-1} \in \mathbb{C}\setminus\mathbb{R}$, *for* $j = 1, \ldots, J$, *and* r_k, $r_k^{-1} \in \mathbb{R}$, *for* $k = 1, \ldots, K$, *and*

$$B(\underline{\omega}) = \sqrt{|a_M| \prod_{k=1}^{K} |r_k|^{-1} \prod_{j=1}^{J} |w_j|^{-2}} \\ \times \prod_{k=1}^{K} \big(\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}} - r_k\big) \prod_{j=1}^{J} \big(\mathrm{e}^{-\mathrm{i}4\pi\underline{\omega}} - 2\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}}\mathrm{Re}\,(w_j) + |w_j|^2\big),$$

is a trigonometric polynomial with real coefficients such that $|B(\underline{\omega})|^2 = A(\underline{\omega})$.

Proof. Let $P_A(\underline{z}) = \sum_{k=-M}^{M} a_k \underline{z}^{k+M}$. This polynomial has $2M$ zeros (counting multiplicities) and since the coefficients are real we have $\overline{P_A(z)} = P_A(\overline{z})$ for all z, so that if z is a zero, then $\overline{z}$ is a zero as well. Moreover, since $a_k = a_{-k}$ for $k = 1, \ldots, M$, it follows that $P_A(\underline{z}) = \underline{z}^{2M} P_A(\frac{1}{\underline{z}})$ and this implies that if z is a zero of P_A, then so is z^{-1}. (The assumption $a_M \neq 0$ guarantees that $P_A(0) \neq 0$.) Moreover, every zero on the unit circle has even multiplicity because $z^{-M} P_A(z)$ is by assumption nonnegative on the unit circle. 1 is a zero, then we see from the relation $P_A(\underline{z}) = \underline{z}^{2M} P_A(\frac{1}{\underline{z}})$ that it is actually a zero of even multiplicity. This gives the conclusion about the zeros of P_A.

Thus we can write P_A in the form

$$P_A(\underline{z}) = a_M \left(\prod_{k=1}^{K} (\underline{z} - r_k)(\underline{z} - r_k^{-1}) \right) \\ \times \left(\prod_{j=1}^{J} (\underline{z} - w_j)(\underline{z} - \overline{w_j})(\underline{z} - w_j^{-1})(\underline{z} - \overline{w_j}^{-1}) \right).$$

Since we for every $z \in \mathbb{C} \setminus 0$ have

$$\begin{aligned}|(\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}} - z)(\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}} - \overline{z}^{-1})| &= |z|^{-1}|(\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}} - z)(\overline{z} - \mathrm{e}^{\mathrm{i}2\pi\underline{\omega}})| \\ &= |z|^{-1}|\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}} - z|^2,\end{aligned}$$

it follows from the nonnegativity of A and the fact that $|A(\underline{\omega})| = |P_A(\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}})|$ that

$$\begin{aligned}A(\underline{\omega}) = |A(\underline{\omega})| = |P_A(\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}})| &= |a_M| \prod_{k=1}^{K} |r_k|^{-1} \prod_{j=1}^{J} |w_j|^{-2} \\ &\times \left| \prod_{k=1}^{K} \left(\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}} - r_k\right) \prod_{j=1}^{J} (\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}} - w_j)(\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}} - \overline{w_j}) \right|^2 = |B(\underline{\omega})|^2.\end{aligned}$$

This completes the proof. □

If we want to construct a wavelet with compact support, the simplest approach according to Theorem 5.5 is to choose a positive integer N, take $L = 0$ for simplicity, since another choice only amounts to a translation, choose the polynomial in (5.9) to be identically zero, and so on.

We leave it as an exercise to show that in this way we get a filter that in addition to (4.11) and (4.46) also satisfies (4.48) and therefore generates a father wavelet or scaling function that turns out to be continuous if $N > 1$. In fact one can say much more about the smoothness of these functions but this question will not be studied here.

4. Properties of compactly supported wavelets

First we consider briefly the question of how one can efficiently calculate the values of the function φ.

Proposition 5.7. *Assume that $(\alpha(k))_{k\in\mathbb{Z}}$ is such that $\alpha(k) = 0$ when $k \leq< a_-$ or $k > a_+$, $\sum_{k=a_-} a_+\alpha(k) = 1$ and $\varphi \in \mathcal{C}_c(\mathbb{R})$, with $\varphi \not\equiv 0$, is a solution to the equation*

$$\varphi(\underline{x}) = 2\sum_{k\in\mathbb{Z}} \alpha(k)\varphi(2\underline{x} - k).$$

Then the matrix A defined by $A(i,j) = 2\alpha(2i - j)$, $i, j = a_- = 1, \ldots, a_+1$ has the eigenvalue 1, $(\varphi(a_- + 1), \ldots, \varphi(a_+1))^{\mathrm{T}}$ is an eigenvector for this eigenvalue and the values of φ at the points $2^{-j}n$, $j \geq 1$ can be recursively calculated from the equation

$$\varphi(2^{-j}n) = 2\sum_{k=a_-}^{a_+} \alpha(k)\varphi(2^{-j+1}n - k), \quad n \in \mathbb{Z}, \quad j \geq 1.$$

Observe that we do not claim that the eigenvalue 1 fro the matrix A has geometric multiplicity 1 so it is may not be clear which eigenvctor to choose, but in most cases this turns out not to be the case.

Our next result restricts the smoothness of the scaling function φ in terms of the support of the filter α.

Theorem 5.8. *If $m \geq 0$ and $f \in C^m(\mathbb{R};\mathbb{C})$, $f \not\equiv 0$, has compact support and satisfies*

$$f = 2\sum_{k=a_-}^{a_+} \alpha(k) f(2\bullet - k), \tag{5.16}$$

for some numbers $\{\alpha(k)\}$, then $m < a_+ - a_- - 1$.

Proof. If we apply Lemma 5.3, we see that the support of f must be contained in the closed interval $[a_-, a_+]$. Thus the support of $f^{(j)}$ must also be contained in this interval for $0 \leq j \leq m$. Moreover, differentiating both sides of (5.16) we get

$$f^{(j)} = 2^{j+1}\sum_{k=a_-}^{a_+} \alpha(k) f^{(j)}(2\bullet - k). \tag{5.17}$$

Let A be a matrix with elements $A(i,j) = 2\alpha_{2i-j}$ for i, $j = a_- + 1, \ldots, a_+ - 1$ (the indexing is nonstandard but this is of no consequence). Now we see from (5.17) that if the vector $(f^{(j)}(a_-+1), f^{(j)}(a_-+2), \ldots, f^{(j)}(a_+-1))^{\mathrm{T}}$ is not the zero vector, then it is an eigenvector of the matrix A corresponding to the eigenvalue 2^{-j}. We leave it as an exercise to show that this vector cannot be the zero vector. Thus A has at least $m+1$ distinct eigenvalues so that A must be at least an $m+1 \times m+1$ matrix. Thus we see that $m + 1 \leq a_+ - a_- - 1$ and this gives the desired conclusion. □

Next we show that except for the Haar function, no father wavelet or scaling function for a multiresolution can not be symmetric with respect to any point.

Proposition 5.9. *Let $(\{V_m\}_{m\in\mathbb{Z}}, \varphi)$ be a multiresolution of $L^2(\mathbb{R};\mathbb{C})$ such that φ is real-valued and has compact support. Then φ is not symmetric (nor antisymmetric) with respect to any point unless φ is the Haar function $\chi_{[0,1]}$.*

Proof. It is clear that we cannot have $\varphi(\lambda + \bullet) = -\varphi(\lambda - \bullet)$ for some $\lambda \in \mathbb{R}$, because then we would have $\int_R \varphi(x)\,\mathrm{d}x = 0$ which is impossible by Theorem 4.9.

Suppose on the other hand that $\varphi(\lambda + \bullet) = \varphi(\lambda - \bullet)$. If λ is an integer, then can take an integer translation of φ, so we may without loss of generality asume that φ is an even function. It follows that the filter α is even as well and has compact support. We shall show that this leads to a contradiction.

Let us introduce the notation that if p is a trigonometric polynomial with period 1, i.e., $p = \sum_k \hat{p}(k)\mathrm{e}^{\mathrm{i}2\pi k\bullet}$, then

$$N_+(p) = \max\{\, k \mid \hat{p}(k) \neq 0 \,\},$$
$$N_-(p) = \min\{\, k \mid \hat{p}(k) \neq 0 \,\}.$$

It is easy to check that

$$N_+(|p|^2) = -N_-(|p|^2) = N_+(p) - N_-(p). \tag{5.18}$$

Let α^e be the sequence $\alpha^e_k = \frac{1}{2}(1 + (-1)^k)\alpha(k)$ with nonzero even indices and α^o the sequence $\alpha^o_k = \frac{1}{2}(1 - (-1)^k)\alpha(k)$ with nonzero odd indices. Since $\widehat{\alpha^e} = \frac{1}{2}(\hat{\alpha}(\bullet) + \hat{\alpha}(\bullet + \frac{1}{2}))$ and $\widehat{\alpha^o} = \frac{1}{2}(\hat{\alpha}(\bullet) - \hat{\alpha}(\bullet + \frac{1}{2}))$, it follows from (4.11) that

$$|\widehat{\alpha^e}|^2 + |\widehat{\alpha^o}|^2 = \tfrac{1}{2}. \tag{5.19}$$

Since neither α^e nor α^o cannot be identically zero (because $\widehat{\alpha^e}(0) = \widehat{\alpha^o}(0) = \frac{1}{2}$) it follows from (5.18) that

$$N_+(\widehat{\alpha^e} - N_+(\widehat{\alpha^e} = N_+(\widehat{\alpha^o} - N_+(\widehat{\alpha^e}. \tag{5.20}$$

From the definition of α^e and α^o we get

$$N_+(\hat{\alpha}) = \max\{N_+(\widehat{\alpha^e}), N_+(\widehat{\alpha^o})\},$$
$$N_-(\hat{\alpha}) = \min\{N_-(\widehat{\alpha^e}), N_-(\widehat{\alpha^o})\},$$

If we combine this result with (5.20) we conclude that

$$N_+(\hat{\alpha}) - N_-(\hat{\alpha}) = \max\{N_+(\widehat{\alpha^e}) - N_-(\widehat{\alpha^o}), N_+(\widehat{\alpha^o}) - N_-(\widehat{\alpha^e})\}. \tag{5.21}$$

Since $N_\pm(\widehat{\alpha^e})$ are even numbers and $N_\pm(\widehat{\alpha^o})$ are odd numbers, it follows that $N_+(\hat{\alpha}) - N_-(\hat{\alpha})$ is an odd number. But then α cannot be an even sequence and we have a contradiction.

Assume next that $\varphi(\lambda + \bullet) = \varphi(\lambda - \bullet)$ where λ is not an integer. We may again shift the function φ so that $\lambda \in (0, 1)$. Taking Fourier transforms we get

$$\hat{\varphi}(\bullet) = \mathrm{e}^{-4\pi\mathrm{i}\lambda\bullet}\hat{\varphi}(-\bullet). \tag{5.22}$$

But then it follows from (4.10) that we also have

$$\hat{\alpha}(\bullet) = \mathrm{e}^{-4\pi\mathrm{i}\lambda\bullet}\hat{\alpha}(-\bullet). \tag{5.23}$$

Now $\hat{\alpha}$ and $\hat{\alpha}(-\bullet)$ are both trigonometric polynomials with period 1 and therefore we must have $\lambda = \frac{1}{2}$. Thus we have $\varphi(\bullet + 1) = \varphi(-\bullet)$. It follows from (4.8) after some changes of variables that $\alpha_{2k+1} = \alpha_{-2k}$ for all $k \in \mathbb{Z}$. Since φ is real-valued, α is real-valued as well, and hence we have

$$\widehat{\alpha^e} = \overline{\widehat{\alpha^o}}, \tag{5.24}$$

and combining this result with (5.19) we get

$$|\widehat{\alpha^e}|^2 = \frac{1}{4}. \tag{5.25}$$

It follows that there exists an index k such that $\alpha_j = \frac{1}{2}$ when $j = 2k+1$ or $j = -2k$. If $k = 0$, then we get the Haar function and otherwise we get

$$\hat{\alpha} = \mathrm{e}^{-\pi\mathrm{i}\bullet} \cos((4k+1)\pi\bullet). \tag{5.26}$$

But then it follows from Proposition 4.15 that (4.48) cannot hold true, and this contradicts Theorem 4.13. $\square$