TKK, Institute of mathematics Gripenberg
Mat-1.3422 Wavelet theory
Exercise 6
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1. Let $N \geq 1$, $P_N(\underline{x}) = \sum_{k=0}^{N-1} \binom{N+k-1}{k} \underline{x}^k$, let α_N be such that $\widehat{\alpha_N}(\underline{\omega}) = (\frac{1}{2}(1+\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}}))^N Q_N(\mathrm{e}^{-\mathrm{i}2\pi\Omega})$ where $|Q_N(\mathrm{e}^{-\mathrm{i}2\pi\Omega})|^2 = P_N(\sin(\pi\omega)^2)$, and let $\psi_N(\underline{x}) = 2\sum_{k\in\mathbb{Z}}(-1)^k \alpha_N(1-k)\varphi(2\underline{x}-k)$ where $\hat{\varphi}(\underline{\omega}) = \prod_{k=1}^{\infty} \hat{\alpha}(2^{-k}\underline{\omega})$. Show that

$$\lim_{N\to\infty} |\widehat{\psi_N}(\omega)| = 0, \quad |\omega| < \frac{1}{2} \quad \text{or} \quad |\omega| > 1.$$

What kind of additional knowledge about $\widehat{\psi_N}$ is needed in order to conlcude that

$$\lim_{N\to\infty} |\widehat{\psi_N}(\omega)| = 1, \quad \frac{1}{2} < |\omega| < 1?$$

2. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Show that if there are two sequnces $(f_n)_{n\in\mathbb{I}}$ and $(g_n)_{n\in\mathbb{I}}$ of elements in H such that

$$\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, f_n \rangle \overline{\langle g, g_n \rangle}, \quad f, g \in H,$$

and

$$\sum_{n=1}^{\infty} \Big(|\langle f, f_n \rangle|^2 + |\langle f, g_n \rangle|^2 \Big) \leq C\|f\|^2, \quad f \in H,$$

for some constant C, then $(f_n)_{n\in\mathbb{I}}$ and $(g_n)_{n\in\mathbb{I}}$ are frames in the space H.

3. Suppose $(f_n)_{n\in\mathbb{I}}$ is a tight frame in H, that is, for same $A > 0$,

$$\sum_{n\in\mathbb{I}} \|\langle f, f_n \rangle\|^2 = A\|f\|^2, \quad f \in H.$$

Show that

$$f = \frac{1}{A} \sum_{n\in\mathbb{I}} \langle f, f_n \rangle f_n, \quad f \in H,$$

by first calculating $A \langle f, g \rangle$ using the formula

$$\langle f, g \rangle = \frac{1}{4}\big(\|f+g\|^2 - \|f-g\|^2 + \mathrm{i}\|f+\mathrm{i}g\|^2 - \mathrm{i}\|f-\mathrm{i}g\|^2\big).$$

4. Let φ be the scaling function and ψ the corresponding wavelet function for an orthonormal multiresolution and assume that φ and ψ have compact support. Let $m \leq 0$ and define

$$\varphi_{\mathbb{T},m,k}(\underline{x}) = \sum_{j\in\mathbb{Z}} 2^{-\frac{m}{2}} \varphi(2^{-m}(\underline{x}+j)-k),$$

$$\psi_{\mathbb{T},m,k}(\underline{x}) = \sum_{j\in\mathbb{Z}} 2^{-\frac{m}{2}} \psi(2^{-m}(\underline{x}+j)-k).$$

Show that if $m_0 \leq 0$ then $\{\, \varphi_{\mathbb{T},m_0,k} \mid k = 0, \ldots, 2^{-m_0}-1 \,\} \cup \{\, \psi_{\mathbb{T},m,k} \mid k = 0, \ldots, 2^{-m}-1, m \leq m_0 \,\}$ is an orthonormal set in $L^2(\mathbb{T}) = L^2([0,1])$.

5. Let φ be the scaling function for an orthonormal multiresolution and assume that φ has compact support. Let $m \leq 0$ and define

$$\varphi_{\mathbb{T},m,k}(\underline{x}) = \sum_{j \in \mathbb{Z}} 2^{-\frac{m}{2}} \varphi(2^{-m}(\underline{x} + j) - k).$$

and let $V_{\mathbb{T},m}$ be the subspace of $L^2(\mathbb{T})$ spanned by $(\varphi_{\mathbb{T},m,k})_{k=0}^{2^{-m}-1}$. Show that $V_m \to L^2(\mathbb{T})$ as $m \to -\infty$.