TKK, Institute of mathematics — Gripenberg
Mat-1.3422 Wavelet theory
Exercise 3
26.9-3.10.2006

1. Let α be a sequence such that $\alpha(k) = 0$ if $k < a_-$ or $k > a_+$. Assume that φ satisfies the equation

$$\varphi(\underline{x}) = 2\sum_{k\in\mathbb{Z}} \alpha(k)\varphi(2\underline{x} - k),$$

and that φ is zero outside some bounded interval. Outside what interval is φ certainly 0?

2. Assume that α is such that for some $N \geq 1$

$$\hat{\alpha}(\underline{\omega}) = \left(\tfrac{1}{2}\left(1 + \mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}}\right)\right)^N Q(\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}}),$$

where

$$\left|Q(\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}})\right|^2 = \sum_{k=0}^{N-1} \binom{N+k-1}{k} \sin(\pi\underline{\omega})^{2k}.$$

Find a set $\mathcal{G}$ which is the finite union of halfopen intervals so that $\sum_{k\in\mathbb{Z}} \chi_{\mathcal{G}}(\omega + k) = 1$ for all $\omega \in \mathbb{R}$ and $\hat{\alpha}$ does not vanish on the set $\overline{\bigcup_{k=1}^{\infty}(2^{-k}\mathcal{G})}$.

3. Let α be the following sequence:

$$\begin{aligned}
\alpha(0) &= \tfrac{1}{8}(1+\sqrt{3}),\\
\alpha(1) &= \tfrac{1}{8}(3+\sqrt{3}),\\
\alpha(2) &= \tfrac{1}{8}(3-\sqrt{3}),\\
\alpha(3) &= \tfrac{1}{8}(1-\sqrt{3}),\\
\alpha(k) &= 0 \quad \text{muuten.}
\end{aligned}$$

Calculate $\sum_{k\in\mathbb{Z}} \alpha(k)$, $\sum_{k\in\mathbb{Z}}(-1)^k\alpha(k)$, $\sum_{k\in\mathbb{Z}}(-1)^k k\alpha(k)$, and $2\sum_{k\in\mathbb{Z}} \alpha(k)\alpha(k+2n)$ when $n \in \mathbb{Z}$. What does this imply for $\hat{\alpha}$?

4. Assume the following POISSON SUMMATION FORMULA to be known: If $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$, the series $\sum_{n=-\infty}^{\infty} f(t+n)$ converges uniformly when $t \in [-\delta, \delta]$ where $\delta > 0$ and the series $\sum_{n=-\infty}^{\infty} \hat{f}(n)$ converges, then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

Show with the aid of this formula that if the function $\varphi \in \mathcal{C}(\mathbb{R})$ has compact support and $n \in \mathbb{N}$ is such that $\hat{\varphi}^{(j)}(m) = 0$ when $m \in \mathbb{Z} \setminus \{0\}$ and $j = 0, 1, \ldots, n-1$ then

$$\sum_{k\in\mathbb{Z}} k^j \varphi(\underline{x} + k) \quad \text{is a polynomial of degree at most } j$$

when $j = 0, 1, \ldots, n-1$.

5. Does the following claim hold: If the function $\varphi \in \mathcal{C}(\mathbb{R})$ has compact support, $\int_{\mathbb{R}} \varphi(t)\,\mathrm{d}t = 1$, $\varphi(\underline{t}) = 2\sum_{k\in\mathbb{Z}} \alpha(k)\varphi(2\underline{t} - k)$, where $\alpha(k) \neq 0$ only for finitely many k and where the the function $\hat{\alpha}$ has a zero of order n in the point $\frac{1}{2}$, then every polynomial of degree at most $n-1$ can be written as a finite linear combination of the functions $\varphi(\underline{t} - k)$ on each bounded interval.