5 Compact spaces

In this section we mainly concentrate on compact Hausdorff spaces, though some results deal with more general classes of topological spaces. Roughly, Hausdorff spaces have enough open sets to distinguish between any two points, while compact spaces "do not have too many open sets". Combining these two properties, compact Hausdorff spaces form an extremely beautiful class to study.

Compact space. Let X be a set and $K \subset X$. A family $\mathcal{S} \subset \mathcal{P}(X)$ is called a *cover of* K if

$$K \subset \bigcup \mathcal{S};$$

if the cover $\mathcal{S}$ is a finite set, it is called a *finite cover*. A cover $\mathcal{S}$ of $K \subset X$ has a *subcover* $\mathcal{S}' \subset \mathcal{S}$ if $\mathcal{S}'$ itself is a cover of K.

Let (X, τ) be a topological space. An *open cover of* X is a cover $\mathcal{U} \subset \tau$ of X. A subset $K \subset X$ is *compact* (more precisely τ*-compact*) if every open cover of K has a finite subcover, i.e.

$$\forall \mathcal{U} \subset \tau \ \exists \mathcal{U}' \subset \mathcal{U}: \ K \subset \bigcup \mathcal{U} \Rightarrow K \subset \bigcup \mathcal{U}' \quad \text{and} \quad |\mathcal{U}'| < \infty.$$

We say that (X, τ) is a compact space if X itself is τ-compact.

Examples.

1. If τ_1 and τ_2 are topologies of X, $\tau_1 \subset \tau_2$, and (X, τ_2) is a compact space then (X, τ_1) is a compact space.

2. $(X, \{\emptyset, X\})$ is a compact space.

3. If $|X| = \infty$ then $(X, \mathcal{P}(X))$ is not a compact space. Clearly any space with a finite topology is compact. Even though a compact topology can be of *any* cardinality, it is in a sense "not far away from being finite".

4. A metric space is compact if and only if it is sequentially compact (i.e. every sequence contains a converging subsequence).

5. A subset $X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded (Heine–Borel Theorem).

6. A theorem due to Frigyes Riesz asserts that a closed ball in a normed vector space over $\mathbb{C}$ (or $\mathbb{R}$) is compact if and only if the vector space is finite-dimensional.

Exercise. A union of two compact sets is compact.

Proposition. *An intersection of a compact set and a closed set is compact.*

Proof. Let $K \subset X$ be a compact set, and $C \subset X$ be a closed set. Let $\mathcal{U}$ be an open cover of $K \cap C$. Then $\{X \setminus C\} \cup \mathcal{U}$ is an open cover of K, thus having a finite subcover $\mathcal{U}'$. Then $\mathcal{U}' \setminus \{X \setminus C\} \subset \mathcal{U}$ is a finite subcover of $K \cap C$; hence $K \cap C$ is compact $\square$

Proposition. *Let X be a compact space and $f : X \to Y$ continuous. Then $f(X) \subset Y$ is compact.*

Proof. Let $\mathcal{V}$ be an open cover of $f(X)$. Then $\mathcal{U} := \{f^{-1}(V) \mid V \in \mathcal{V}\}$ is an open cover of X, thus having a finite subcover $\mathcal{U}'$. Hence $f(X)$ is covered by $\{f(U) \mid U \in \mathcal{U}'\} \subset \mathcal{V}$ $\square$

Corollary. *If X is compact and $f \in C(X)$ then $|f|$ attains its greatest value on X (here $|f|(x) := |f(x)|$)* $\square$

5.1 Compact Hausdorff spaces

Theorem. *Let X be a Hausdorff space, $A, B \subset X$ compact subsets, and $A \cap B = \emptyset$. Then there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. (In particular, compact sets in a Hausdorff space are closed.)*

Proof. The proof is trivial if $A = \emptyset$ or $B = \emptyset$. So assume $x \in A$ and $y \in B$. Since X is a Hausdorff space and $x \neq y$, we can choose neighborhoods $U_{xy} \in \mathcal{V}(x)$ and $V_{xy} \in \mathcal{V}(y)$ such that $U_{xy} \cap V_{xy} = \emptyset$. The collection $\mathcal{P} = \{V_{xy} \mid y \in B\}$ is an open cover of the compact set B, so that it has a finite subcover

$$\mathcal{P}_x = \{V_{xy_j} \mid 1 \leq j \leq n_x\} \subset \mathcal{P}$$

for some $n_x \in \mathbb{N}$. Let

$$U_x := \bigcap_{j=1}^{n_x} U_{xy_j}.$$

Now $\mathcal{O} = \{U_x \mid x \in A\}$ is an open cover of the compact set A, so that it has a finite subcover

$$\mathcal{O}' = \{U_{x_i} \mid 1 \leq i \leq m\} \subset \mathcal{O}.$$

Then define

$$U := \bigcup \mathcal{O}', \quad V := \bigcap_{i=1}^{m} \bigcup \mathcal{P}_{x_i}.$$

It is an easy task to check that U and V have desired properties $\square$

Corollary. *Let X be a compact Hausdorff space, $x \in X$, and $W \in \mathcal{V}(x)$. Then there exists $U \in \mathcal{V}(x)$ such that $\overline{U} \subset W$.*

Proof. Now $\{x\}$ and $X \setminus W$ are closed sets in a compact space, thus they are compact. Since these sets are disjoint, there exist open disjoint sets $U, V \subset X$ such that $x \in U$ and $X \setminus W \subset V$; i.e.

$$x \in U \subset X \setminus V \subset W.$$

Hence $x \in U \subset \overline{U} \subset X \setminus V \subset W$ $\square$

Proposition. *Let (X, τ_X) be a compact space and (Y, τ_Y) a Hausdorff space. A bijective continuous mapping $f : X \to Y$ is a homeomorphism.*

Proof. Let $U \in \tau_X$. Then $X \setminus U$ is closed, hence compact. Consequently, $f(X \setminus U)$ is compact, and due to the Hausdorff property $f(X \setminus U)$ is closed. Therefore $(f^{-1})^{-1}(U) = f(U)$ is open $\square$

Corollary. *Let X be a set with a compact topology τ_2 and a Hausdorff topology τ_1. If $\tau_1 \subset \tau_2$ then $\tau_1 = \tau_2$.*

Proof. The identity mapping $(x \mapsto x) : X \to X$ is a continuous bijection from (X, τ_2) to (X, τ_1) $\square$

A more direct proof of the Corollary. Let $U \in \tau_2$. Since (X, τ_2) is compact and $X \setminus U$ is τ_2-closed, $X \setminus U$ must be τ_2-compact. Now $\tau_1 \subset \tau_2$, so that $X \setminus U$ is τ_1-compact. (X, τ_1) is Hausdorff, implying that $X \setminus U$ is τ_1-closed, thus $U \in \tau_1$; this yields $\tau_2 \subset \tau_1$ $\square$

Functional separation

A family $\mathcal{F}$ of mappings $X \to \mathbb{C}$ is said to *separate the points of the set X* if there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$ whenever $x \neq y$. Later in these notes we shall discover that a compact space X is metrizable if and only if

$C(X)$ is separable and separates the points of X.

Urysohn's Lemma is the key result of this section:

Urysohn's Lemma (1923?). *Let X be a compact Hausdorff space, $A, B \subset X$ closed non-empty sets, $A \cap B = \emptyset$. Then there exists $f \in C(X)$ such that*

$$0 \leq f \leq 1, \quad f(A) = \{0\}, \quad f(B) = \{1\}.$$

Proof. The set $\mathbb{Q} \cap [0,1]$ is countably infinite; let $\phi : \mathbb{N} \to \mathbb{Q} \cap [0,1]$ be a bijection satisfying $\phi(0) = 0$ and $\phi(1) = 1$. Choose open sets $U_0, U_1 \subset X$ such that

$$A \subset U_0 \subset \overline{U_0} \subset U_1 \subset \overline{U_1} \subset X \setminus B.$$

Then we proceed inductively as follows: Suppose we have chosen open sets $U_{\phi(0)}, U_{\phi(1)}, \ldots, U_{\phi(n)}$ such that

$$\phi(i) < \phi(j) \Rightarrow \overline{U_{\phi(i)}} \subset U_{\phi(j)}.$$

Let us choose an open set $U_{\phi(n+1)} \subset X$ such that

$$\phi(i) < \phi(n+1) < \phi(j) \Rightarrow \overline{U_{\phi(i)}} \subset U_{\phi(n+1)} \subset \overline{U_{\phi(n+1)}} \subset U_{\phi(j)}$$

whenever $0 \leq i, j \leq n$. Let us define

$$r < 0 \Rightarrow U_r := \emptyset, \quad s > 1 \Rightarrow U_s := X.$$

Hence for each $q \in \mathbb{Q}$ we get an open set $U_q \subset X$ such that

$$\forall r, s \in \mathbb{Q} : \ r < s \Rightarrow \overline{U_r} \subset U_s.$$

Let us define a function $f : X \to [0,1]$ by

$$f(x) := \inf\{r : \ x \in U_r\}.$$

Clearly $0 \leq f \leq 1$, $f(A) = \{0\}$ and $f(B) = \{1\}$.

Let us prove that f is continuous. Take $x \in X$ and $\varepsilon > 0$. Take $r, s \in \mathbb{Q}$ such that

$$f(x) - \varepsilon < r < f(x) < s < f(x) + \varepsilon;$$

then f is continuous at x, since $x \in U_s \setminus \overline{U_r}$ and for every $y \in U_s \setminus \overline{U_r}$ we have $|f(y) - f(x)| < \varepsilon$. Thus $f \in C(X)$ $\square$

Corollary. *Let X be a compact space. Then $C(X)$ separates the points of X if and only if X is Hausdorff.*

Exercise. Prove the previous Corollary.