10 C*-algebras

Now we are finally in the position to abstractly characterize algebras $C(X)$ among Banach algebras: according to Gelfand and Naimark, the category of compact Hausdorff spaces is equivalent to the category of commutative C*-algebras. The class of C*-algebras behaves nicely, and the related functional analysis adequately deserves the name "non-commutative topology".

Involutive algebra. An algebra $\mathcal{A}$ is a $*$-*algebra* ("*star-algebra*" or an *involutive algebra*) if there is a mapping $(x \mapsto x^*) : \mathcal{A} \to \mathcal{A}$ satisfying

$$(\lambda x)^* = \overline{\lambda} x^*, \quad (x+y)^* = x^* + y^*, \quad (xy)^* = y^* x^*, \quad (x^*)^* = x$$

for every $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$; such a mapping is called an *involution*. In other words, an involution is a conjugate-linear anti-multiplicative self-invertible mapping $\mathcal{A} \to \mathcal{A}$.

A $*$-*homomorphism* $\phi : \mathcal{A} \to \mathcal{B}$ between involutive algebras $\mathcal{A}$ and $\mathcal{B}$ is an algebra homomorphism satisfying

$$\phi(x^*) = \phi(x)^*$$

for every $x \in \mathcal{A}$. The set of all $*$-homomorphisms between $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathrm{Hom}^*(\mathcal{A}, \mathcal{B})$.

C*-algebra. A *C*-algebra* $\mathcal{A}$ is an involutive Banach algebra such that

$$\|x^* x\| = \|x\|^2$$

for every $x \in \mathcal{A}$.

Examples.

1. The Banach algebra $\mathbb{C}$ is a C*-algebra with the involution $\lambda \mapsto \lambda^* = \overline{\lambda}$, i.e. the complex conjugation.

2. If K is a compact space then $C(K)$ is a commutative C*-algebra with the involution $f \mapsto f^*$ by complex conjugation, $f^*(x) := \overline{f(x)}$.

3. $L^\infty([0,1])$ is a C*-algebra, when the involution is as above.

4. $A(\mathbb{D}(0,1)) = C\left(\overline{\mathbb{D}(0,1)}\right) \cap H(\mathbb{D}(0,1))$ is an involutive Banach algebra with $f^*(z) := \overline{f(\overline{z})}$, but it is not a C*-algebra.

5. The radical of a commutative C*-algebra is always the trivial $\{0\}$, and thus 0 is the only nilpotent element. Hence for instance the algebra of matrices $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ (where $\alpha, \beta \in \mathbb{C}$) cannot be a C*-algebra.

6. If $\mathcal{H}$ is a Hilbert space then $\mathcal{L}(\mathcal{H})$ is a C*-algebra when the involution is the usual adjunction $A \mapsto A^*$, and clearly any norm-closed involutive subalgebra of $\mathcal{L}(\mathcal{H})$ is also a C*-algebra. Actually, there are no others, but we shall not prove this fact in these lecture notes:

Gelfand–Naimark Theorem (1943). *If $\mathcal{A}$ is a C*-algebra then there exists a Hilbert space $\mathcal{H}$ and an isometric $*$-homomorphism onto a closed involutive subalgebra of $\mathcal{L}(\mathcal{H})$* $\square$

However, we shall characterize the commutative case: the Gelfand transform of a commutative C*-algebra $\mathcal{A}$ will turn out to be an isometric isomorphism $\mathcal{A} \to C(\mathrm{Spec}(\mathcal{A}))$, so that $\mathcal{A}$ "is" the function algebra $C(K)$ for the compact Hausdorff space $K = \mathrm{Spec}(\mathcal{A})$! Before going into this, we prove some related results.

Proposition. *Let $\mathcal{A}$ be a $*$-algebra. Then $\mathbb{I}^* = \mathbb{I}$, $x \in \mathcal{A}$ is invertible if and only if $x^* \in \mathcal{A}$ is invertible, and $\sigma(x^*) = \overline{\sigma(x)} := \{\overline{\lambda} \mid \lambda \in \sigma(x)\}$.*

Proof. First,

$$\mathbb{I}^* = \mathbb{I}^*\mathbb{I} = \mathbb{I}^*(\mathbb{I}^*)^* = (\mathbb{I}^*\mathbb{I})^* = (\mathbb{I}^*)^* = \mathbb{I};$$

second,

$$(x^{-1})^*x^* = (xx^{-1})^* = \mathbb{I}^* = \mathbb{I} = \mathbb{I}^* = (x^{-1}x)^* = x^*(x^{-1})^*;$$

third,

$$\overline{\lambda}\mathbb{I} - x^* = (\lambda\mathbb{I}^*)^* - x^* = (\lambda\mathbb{I})^* - x^* = (\lambda\mathbb{I} - x)^*,$$

which concludes the proof $\square$

Proposition. *Let $\mathcal{A}$ be a C*-algebra, and $x = x^* \in \mathcal{A}$. Then $\sigma(x) \subset \mathbb{R}$.*

Proof. **Assume** that $\lambda \in \sigma(x) \setminus \mathbb{R}$, i.e. $\lambda = \lambda_1 + i\lambda_2$ for some $\lambda_j \in \mathbb{R}$ with $\lambda_2 \neq 0$. Hence we may define $y := (x - \lambda_1\mathbb{I})/\lambda_2 \in \mathcal{A}$. Now $y^* = y$. Moreover, $i \in \sigma(y)$, because

$$i\mathbb{I} - y = \frac{\lambda\mathbb{I} - x}{\lambda_2}.$$

Take $t \in \mathbb{R}$. Then $t+1 \in \sigma(t\mathbb{I} - iy)$, because

$$(t+1)\mathbb{I} - (t\mathbb{I} - iy) = -i(i\mathbb{I} - y).$$

Thereby

$$\begin{array}{rcl} (t+1)^2 & \leq & \rho(t\mathbb{I} - iy)^2 \\ & \leq & \|t\mathbb{I} - iy\|^2 \\ & \overset{\mathrm{C}^*}{=} & \|(t\mathbb{I} - iy)^*(t\mathbb{I} - iy)\| \\ & \overset{t\in\mathbb{R},\ y^*=y}{=} & \|(t\mathbb{I} + iy)(t\mathbb{I} - iy)\| \ = \ \|t^2\mathbb{I} + y^2\| \\ & \leq & t^2 + \|y\|, \end{array}$$

so that $2t + 1 \leq \|y\|$ for every $t \in \mathbb{R}$; a **contradiction** $\square$

Corollary. *Let $\mathcal{A}$ a C^*-algebra, $\phi : \mathcal{A} \to \mathbb{C}$ a homomorphism, and $x \in \mathcal{A}$. Then $\phi(x^*) = \overline{\phi(x)}$, i.e. ϕ is a $*$-homomorphism.*

Proof. Define the "real part" and the "imaginary part" of x by

$$u := \frac{x + x^*}{2}, \quad v := \frac{x - x^*}{2i}.$$

Then $x = u + iv$, $u^* = u$, $v^* = v$, and $x^* = u - iv$. Since a homomorphism maps invertibles to invertibles, we have $\phi(u) \in \sigma(u)$; we know that $\sigma(u) \subset \mathbb{R}$, because $u^* = u$. Similarly we obtain $\phi(v) \in \mathbb{R}$. Thereby

$$\phi(x^*) = \phi(u - iv) = \phi(u) - i\phi(v) = \overline{\phi(u) + i\phi(v)} = \overline{\phi(u + iv)} = \overline{\phi(x)};$$

this means that $\mathrm{Hom}^*(\mathcal{A}, \mathbb{C}) = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$ $\square$

Exercise. Let $\mathcal{A}$ be a Banach algebra, $\mathcal{B}$ its closed subalgebra, and $x \in \mathcal{B}$. Prove the following facts:
(a) $G(\mathcal{B})$ is open and closed in $G(\mathcal{A}) \cap \mathcal{B}$.
(b) $\sigma_{\mathcal{A}}(x) \subset \sigma_{\mathcal{B}}(x)$ and $\partial\sigma_{\mathcal{B}}(x) \subset \partial\sigma_{\mathcal{A}}(x)$.
(c) If $\mathbb{C} \setminus \sigma_{\mathcal{A}}(x)$ is connected then $\sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{B}}(x)$.

Using the results of the exercise above, the reader can prove the following important fact on the invariance of spectrum in C*-algebras:

Exercise*. Let $\mathcal{A}$ be a C*-algebra and $\mathcal{B}$ its C*-subalgebra. Show that $\sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{A}}(x)$ for every $x \in \mathcal{B}$.

Lemma. *Let $\mathcal{A}$ be a C*-algebra. Then $\|x\|^2 = \rho(x^*x)$ for every $x \in \mathcal{A}$.*

Proof. Now

$$\|(x^*x)^2\| = \|(x^*x)(x^*x)\| = \|(x^*x)^*(x^*x)\| \overset{\mathrm{C}^*}{=} \|x^*x\|^2,$$

so that by induction

$$\|(x^*x)^{2^n}\| = \|x^*x\|^{2^n}$$

for every $n \in \mathbb{N}$. Therefore applying the Spectral Radius Formula, we get

$$\rho(x^*x) = \lim_{n\to\infty} \|(x^*x)^{2^n}\|^{1/2^n} = \lim_{n\to\infty} \|x^*x\|^{2^n/2^n} = \|x^*x\|,$$

the result we wanted □

Exercise*. Let $\mathcal{A}$ be a C*-algebra. Show that there can be at most one C*-algebra norm on an involutive Banach algebra. Moreover, prove that if $\mathcal{A}, \mathcal{B}$ are C*-algebras then $\phi \in \mathrm{Hom}^*(\mathcal{A}, \mathcal{B})$ is continuous and has a norm $\|\phi\| = 1$.

Commutative Gelfand–Naimark. *Let $\mathcal{A}$ be a commutative C*-algebra. Then the Gelfand transform $(x \mapsto \widehat{x}) : \mathcal{A} \to C(\mathrm{Spec}(\mathcal{A}))$ is an isometric $*$-isomorphism.*

Proof. Let $K = \mathrm{Spec}(\mathcal{A})$. We already know that the Gelfand transform is a Banach algebra homomorphism $\mathcal{A} \to C(K)$. Let $x \in \mathcal{A}$ and $\phi \in K$. Since ϕ is actually a $*$-homomorphism, we get

$$\widehat{x^*}(\phi) = \phi(x^*) = \overline{\phi(x)} = \overline{\widehat{x}(\phi)} = \widehat{x}^*(\phi);$$

the Gelfand transform is a $*$-homomorphism.

Now we have proven that $\widehat{\mathcal{A}} \subset C(K)$ is an involutive subalgebra separating the points of K. Stone–Weierstrass Theorem thus says that $\widehat{\mathcal{A}}$ is dense in $C(K)$. If we can show that the Gelfand transform $\mathcal{A} \to \widehat{\mathcal{A}}$ is an isometry then we must have $\widehat{\mathcal{A}} = C(K)$: Take $x \in \mathcal{A}$. Then

$$\|\widehat{x}\|^2 = \|\widehat{x}^*\widehat{x}\| = \|\widehat{x^*x}\| \overset{\text{Gelfand}}{=} \rho(x^*x) \overset{\text{Lemma}}{=} \|x\|^2,$$

i.e. $\|\widehat{x}\| = \|x\|$ □

Exercise*. Show that an injective $*$-homomorphism between C*-algebras is an isometry. (Hint: Gelfand transform.)

Exercise*. A linear functional f on a C*-algebra $\mathcal{A}$ is called *positive* if $f(x^*x) \geq 0$ for every $x \in \mathcal{A}$. Show that the positive functionals separate the points of $\mathcal{A}$.

Exercise*. Prove that the involution of a C*-algebra cannot be altered without destroying the C*-property $\|x^*x\| = \|x\|^2$.

An element x of a C*-algebra is called *normal* if $x^*x = xx^*$. We use the commutative Gelfand–Naimark Theorem to create the so called continuous functional calculus at a normal element — a non-commutative C*-algebra admits some commutative studies:

Theorem. *Let $\mathcal{A}$ be a C*-algebra, and $x \in \mathcal{A}$ be a normal element. Let $\iota = (\lambda \mapsto \lambda) : \sigma(x) \to \mathbb{C}$. Then there exists a unique isometric $*$-homomorphism $\phi : C(\sigma(x)) \to \mathcal{A}$ such that $\phi(\iota) = x$ and $\phi(C(\sigma(x)))$ is the C*-algebra generated by x, i.e. the smallest C*-algebra containing $\{x\}$.*

Proof. Let $\mathcal{B}$ be the C*-algebra generated by x. Since x is normal, $\mathcal{B}$ is commutative. Let $\mathrm{Gel} = (y \mapsto \widehat{y}) : \mathcal{B} \to C(\mathrm{Spec}(\mathcal{B}))$ be the Gelfand transform of $\mathcal{B}$. The reader may easily verify that

$$\widehat{x} : \mathrm{Spec}(\mathcal{B}) \to \sigma(x)$$

is a continuous bijection from a compact space to a Hausdorff space; hence it is a homeomorphism. Let us define the mapping

$$C_{\widehat{x}} : C(\sigma(x)) \to C(\mathrm{Spec}(\mathcal{B})), \quad (C_{\widehat{x}} f)(h) := f(\widehat{x}(h)) = f(h(x));$$

$C_{\widehat{x}}$ can be thought as a "transpose" of $\widehat{x}$. Let us define

$$\phi = \mathrm{Gel}^{-1} \circ C_{\widehat{x}} : C(\sigma(x)) \to \mathcal{B} \subset \mathcal{A}.$$

Then $\phi : C(\sigma(x) \to \mathcal{A}$ is obviously an isometric $*$-homomorphism. Furthermore,

$$\phi(\iota) = \mathrm{Gel}^{-1}(C_{\widehat{x}}(\iota)) = \mathrm{Gel}^{-1}(\widehat{x}) = \mathrm{Gel}^{-1}(\mathrm{Gel}(x)) = x.$$

Due to the Stone–Weierstrass Theorem, the $*$-algebra generated by $\iota \in C(\sigma(x))$ is dense in $C(\sigma(x))$; since the $*$-homomorphism ϕ maps the generator ι to the generator x, the uniqueness of ϕ follows $\square$

Remark. The $*$-homomorphism $\phi : C(\sigma(x)) \to \mathcal{A}$ in above is called the *(continuous) functional calculus at the normal element* $\phi(\iota) = x \in \mathcal{A}$. If $p = (z \mapsto \sum_{j=1}^{n} a_j z^j) : \mathbb{C} \to \mathbb{C}$ is a polynomial then it is natural to define $p(x) := \sum_{j=1}^{n} a_j x^j$. Then actually

$$p(x) = \phi(p);$$

hence it is natural to define $f(x) := \phi(f)$ for every $f \in C(\sigma(x))$. It is easy to check that if $f \in C(\sigma(x))$ and $h \in \mathrm{Spec}(\mathcal{B})$ then $f(h(x)) = h(f(x))$.

Exercise. Let $\mathcal{A}$ be a C*-algebra, $x \in \mathcal{A}$ normal, $f \in C(\sigma(x))$, and $g \in C(f(\sigma(x)))$. Show that $\sigma(f(x)) = f(\sigma(x))$ and that $(g \circ f)(x) = g(f(x))$.