Mat-1.3608 Markov chains. Recall the following notation from Brémaud: π is a probability distribution on $S = (s_1, , \ldots, s_k)$ such that $\pi_i > 0$ for all $i = 1, \ldots, k$. Then for $x, y \in \mathbb{R}^k$ put $< x, y >_\pi = \sum_{i=1}^k x_i y_i \pi_i$.

V Exercise 21.2. 2008 Tikanmäki/Valkeila.

1. Let P be a transition matrix. We say that it is *doubly stochastic* if
$$\sum_j p_{ij} = 1 \quad \text{for fixed} \quad i \quad \text{and} \quad \sum_i p_{ij} = 1 \quad \text{for a fixed} \quad j.$$
Give an example of doubly stochastic matrix P which is not non-negative definite.
2. Let P be the matrix
$$\begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$
for $\alpha, \beta \in (0,1)$. Compute P^n in terms of the spectral repserentation.
3. [Continuation] Compute the stationary distribution π and the difference $P^n - \mathbf{1}\pi^T$.
4. Show that (P, π) is reversible if and only if $< Px, y >_\pi = < x, Py >_\pi$.
5. Let α, β be two probability measures on S. Put
$$d_V(\alpha, \beta) = \frac{1}{2}|\alpha - \beta| = \frac{1}{2}\sum_{i=1}^k |\alpha_i - \beta_i|.$$
Show that
$$d_V(\alpha, \beta) = \frac{1}{2} \sup_{y \in \mathbb{R}^k} \left(\sum_{i=1}^k \alpha_i y_i - \sum_{i=1}^k \beta_i y_i : \sup_i |y_i| = 1 \right).$$
6. Define the χ^2- contrast by
$$\chi^2(\alpha; \beta) = \sum_i \frac{(\alpha_i - \beta_i)^2}{\beta_i}.$$
Show that
$$\chi^2(\alpha; \pi) = ||\alpha - \pi||^2_{\frac{1}{\pi}}.$$