1. We use the notation

$$\pi(+ \mid \mathrm{m}) = \text{prob. of positive mammogram with a malignant tumor,}$$

and similarly with the other cases. From the table,

$$\pi(+ \mid \mathrm{m}) = 0.8, \quad \pi(- \mid \mathrm{m}) = 0.2,$$

$$\pi(+ \mid \mathrm{b}) = 0.1, \quad \pi(- \mid \mathrm{b}) = 0.9,$$

The subjective probabilities of having malignant/beningn tumor, based on the patient's consulting the internet, are

$$\pi(\mathrm{m}) = 0.01, \quad \pi(\mathrm{b}) = 0.99.$$

Notice: these numbers are not so reliable, since the probability of having a lump in the brest is not taken into account, and one should indeed condition on that.

(a) Probability of positive mammogram result:

$$\begin{aligned} \pi(+) &= \pi(+ \mid \mathrm{m})\pi(\mathrm{m}) + \pi(+ \mid \mathrm{b})\pi(\mathrm{b}) \\ &= 0.8 \cdot 0.01 + 0.1 \cdot 0.99 \\ &= 0,107 \\ &\approx 0,1. \end{aligned}$$

(b) Conditional probability of the malignant tumor, conditioned on the fact that the mammogram was positive, by Bayes formula:

$$\begin{aligned} \pi(\mathrm{m} \mid +) &= \frac{\pi(+ \mid \mathrm{m})\pi(\mathrm{m})}{\pi(+)} \\ &\approx 0,07. \end{aligned}$$

2. Your prior probability for believing the story

$$\pi(+) = x, \quad \pi(-) = 1 - x.$$

Conditional probabilities: If the guy makes a wild guess, i.e., there is no gift,

$$\pi(\text{right} \mid -) = 0.01, \quad \pi(\text{wrong} \mid -) = 0.99.$$

The guy claims to have the gift, and gives the conditional probabilities of his success:

$$\pi(\text{right} \mid +) = 0.8, \quad \pi(\text{wrong} \mid -) = 0.2.$$

In the light of this, your subjective probability for his success of getting it right is

$$\begin{aligned}\pi(\text{right}) &= \pi(\text{right} \mid +)\pi(+) + \pi(\text{right} \mid -)\pi(-) \\ &= 0.8x + 0.01(1-x).\end{aligned}$$

By Bayes formyla, the probability of the claimed gift, considering the fact that the guy got it right, is

$$\begin{aligned}\pi(+ \mid \text{right}) &= \frac{\pi(\text{right} \mid +)\pi(+)}{\pi(\text{right})} \\ &= \frac{0,8x}{0,8x + 0,01(1-x)}.\end{aligned}$$

Set

$$\pi(+ \mid \text{right}) < 0.5,$$

and solve the bound for x.

3. Divide 24 hours in n intervals Δ_j, the length of Δ_j being t_j (hours). The probability density of your waiting time, assuming that you arrive during Δ_j to the station, is

$$\pi(t \mid \Delta_j) = \frac{1}{t_j}\chi_{\Delta_j}(t).$$

The conditional expectation of your waiting time is then

$$\mathrm{E}\{T \mid \Delta_j\} = \int t\pi(t \mid \Delta_j)dt = \frac{t_j}{2}.$$

The probability to arrive to the station during Δ_j is

$$\pi(\Delta_j) = \frac{t_j}{24},$$

so the average waiting time is

$$\mathrm{E}\{T\} = \sum_{j=1}^{n} \mathrm{E}\{T \mid \Delta_j\}\pi(\Delta_j) =$$

4. Take a set $B \in \mathbb{R}_+$ and denote

$$\widetilde{B} = \big\{(x_1, x_2) \mid r = \sqrt{x_1^2 + x_2^2} \in B\big\} \subset \mathbb{R}^2.$$

The componets X_1 and X_2 are independent, so the joint density is

$$\pi(x_1, x_2) = \pi(x_1)\pi(x_2) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{1}{2\sigma^2}(x_1^2 + x_2^2)\right).$$

We have

$$\begin{aligned}
\mathrm{P}\big\{R \in B\big\} &= \mathrm{P}\big\{(X_1, X_2) \in \widetilde{B}\big\} = \int_{\widetilde{B}} \pi(x_1, x_2) dx_1 dx_2 \\
&= \frac{1}{2\pi\sigma^2}\int_B \int_0^{2\pi} \exp\left(-\frac{1}{2\sigma^2}\underbrace{(x_1^2 + x_2^2)}_{=r^2}\right) d\theta r dr \\
&= \int_B \frac{r}{\sigma^2}\exp\left(-\frac{1}{2\sigma^2}r^2\right) dr,
\end{aligned}$$

so the Rayleigh distribution is given by the density

$$\pi(r) = \frac{r}{\sigma^2}\exp\left(-\frac{1}{2\sigma^2}r^2\right).$$

5. For moments of the Gaussian random variable $X \sim \mathcal{N}(\mu, \sigma)$ we see with the change of variable $x = \psi(y) = y + \mu$ that[1]

$$\begin{aligned}
E\{(X - \overline{x})^k\} &= \left(\frac{1}{2\pi\sigma}\right)^{\frac{1}{2}} \int_{\mathbb{R}} (x - \mu)^k e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&\overset{\text{CoV}}{=} \left(\frac{1}{2\pi\sigma}\right)^{\frac{1}{2}} \int_{\mathbb{R}} y^k e^{-\frac{y}{2\sigma^2}} dy.
\end{aligned}$$

Hence the skewness of X is

$$\left(\frac{1}{2\pi\sigma}\right)^{\frac{1}{2}} \int_{\mathbb{R}} y^3 e^{-\frac{y}{2\sigma^2}} dy = 0,$$

since the integrad is an odd function (and integrable).

[1] CoV = Change of Variables

For the kurtosis we use partial integration

$$
\begin{aligned}
E\{(X-\overline{x})^k\} &= \left(\frac{1}{2\pi\sigma}\right)^{\frac{1}{2}} \int_{\mathbb{R}} y^4 e^{-\frac{y}{2\sigma^2}}\, dy \\
&\overset{\text{PI}}{=} \left(\frac{1}{2\pi\sigma}\right)^{\frac{1}{2}} \left(\underbrace{\Big/_{-\infty}^{\infty} - y^3\sigma^2 e^{-\frac{y^2}{2\sigma^2}}}_{0} + \int_{-\infty}^{\infty} 3y^2\sigma^2 e^{-\frac{y}{2\sigma^2}}\, dy \right) \\
&\overset{\text{PI}}{=} \left(\frac{1}{2\pi\sigma}\right)^{\frac{1}{2}} \left(\underbrace{\Big/_{-\infty}^{\infty} - 3\sigma^4 e^{-\frac{y^2}{2\sigma^2}}}_{0} + \int_{-\infty}^{\infty} 3\sigma^4 e^{-\frac{y}{2\sigma^2}}\, dy \right) \\
&= 3\sigma^4, \qquad (1)
\end{aligned}
$$

since[2] the integral of the Gaussian over $\mathbb{R}$ is 1.

[2] PI = Partial Integration