Computational Methods in Inverse Problems, Mat–1.3626

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Let f be a real function defined over the interval $[0, \infty)$. The *Laplace transform* $\mathcal{L}f$ of f is defined as the integral

$$\mathcal{L}f(s) = \int_0^\infty e^{-st} f(t)dt,$$

provided that the integral is convergent. We consider the following problem: Given the values of the Laplace transform at points s_j, $0 < s_1 < \ldots < s_n < \infty$, we want to estimate the function f. To this end, we approximate first the integral defining the Laplace transform by a finite sum,

$$\int_0^\infty e^{-s_j t} f(t)dt \approx \sum_{k=1}^n w_k e^{-s_j t_k} f(t_k),$$

where, w_k's are the weights and t_k's are the nodes of the quadrature rule, e.g., Gauss quadrature, Simpson's rule or the trapezoid rule. Let $x_k = f(t_k)$, $y_j = \mathcal{L}f(s_j)$ and $a_{jk} = w_k e^{-s_j t_k}$, and write the numerical approximation of the Laplace transform in the form $Ax = y$, where A is an $n \times n$ square matrix. In this example, choose the data points logarithmically distributed, e.g.,

$$\log(s_j) = (-1 + \frac{j-1}{20}) \log 10, \quad 1 \le j \le 40$$

to guarantee denser sampling near the origin.

Use a quadrature rule of your choice, with 40 nodes t_k in the interval $[0, 5]$. Hence, $A \in \mathbb{R}^{40\times 40}$.

To generate the data, let the function f, or true signal, be

$$f(t) = \begin{cases} t, & \text{if } 0 \le t < 1, \\ \frac{3}{2} - \frac{1}{2}t, & \text{if } 1 \le t < 3, \\ 0, & \text{if } t \ge 3, \end{cases}$$

The Laplace transform can then be calculated analytically. We have

$$\mathcal{L}f(s) = \frac{1}{2s^2}(2 - 3e^{-s} + e^{-3s}).$$

(Check this, show details).

To appreciate the ill-posedness of this problem, try to estimate the values $x_j = f(t_j)$ by direct solution of the system $Ax = y$, using the "backslash" command in Matlab, using analytically known data with no artificial error added to it.

Calculate the SVD of A and show the singular values.

Add a tiny random error to the data, and estimate x from the noisy data using the Tikhonov regularization,

$$x_\delta = \text{argmin}\big(\|Ax - y\|^2 + \delta^2\|x\|^2\big).$$

Try different values of the regularization parameter, and calculate the corresponding discrepancies. Show the estimates with different values of δ.