Computational Methods in Inverse Problems, Mat–1.3626

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Excercise 2, 1.10.–7.10.2007

1. Consider the equation
$$Ax = b, \quad A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m.$$
By multiplying the equation from the left with A^{T} we obtain the so called *normal equations*,
$$A^{\mathrm{T}}Ax = A^{\mathrm{T}}b,$$
where the matrix $A^{\mathrm{T}}A \in \mathbb{R}^{n\times n}$ is a square matrix, so the system is formally determined. Analyze the normal equations using the singular value decomposition of A. In particular, answer the following questions: In terms of the singular values of A, when is $A^{\mathrm{T}}A$ invertible? What is the pseudoniverse of it? What is the connection between the pseudoinverse of A and $(A^{\mathrm{T}}A)^\dagger A^{\mathrm{T}}$.

2. In terms of the singular values of A, when is AA^{T} invertible? Is $A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b$ then a solution of $Ax = b$? What about $A^{\mathrm{T}}(AA^{\mathrm{T}})^\dagger$? Is it equal to $(A^{\mathrm{T}}A)^\dagger A^{\mathrm{T}}b$, and if not in general, under what conditions?

3. The mappings $A^\dagger A$ and $AA^\dagger$ are orthogonal projections. Show this, and find the subspaces on which they project.

4. Show the *Moore-Penrose identities*,
$$\begin{aligned} A^\dagger AA^\dagger &= A^\dagger, \\ AA^\dagger A &= A, \\ (A^\dagger A)^{\mathrm{T}} &= A^\dagger A, \\ (AA^\dagger)^{\mathrm{T}} &= AA^\dagger. \end{aligned}$$
Can you interpret these results?

5. Normal equations and ill-conditioning: Form the matrix $A \in \mathbb{R}^{2\times 2}$,
$$A = UDU^{\mathrm{T}},$$
where
$$U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad D = \begin{bmatrix} 1 & \\ & 10^{-k} \end{bmatrix},$$
where $\theta = \pi/3$, and $k = 10$, for instance. Using Matlab, calculate
$$A^{-1}b, \quad \text{and} \quad (A^{\mathrm{T}}A)^{\mathrm{T}}A^{\mathrm{T}}b, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
Why are they not equal? Change the value k and analyze what happens.