Mat-5.3741 Theory of Elasticity (5 cp) L **Spring 2007**

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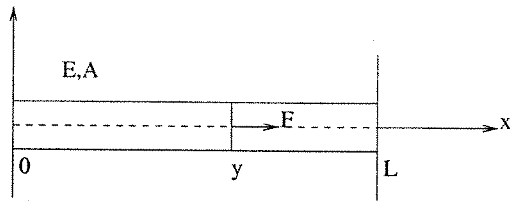
Exercise 1

Problem 1

If you do not remember (or have not seen) the result $\mathrm{N}(B) = \mathrm{R}(B^T)^\perp$ from linear algebra, prove it (or look it up in your textbook).

Problem 2

Consider a rod (E, A, L) fixed at both ends and loaded by a force F at point y, $(0 < y < L)$:



Find the expression for the displacement $u(x)$ in the form

$$u(x) = K(x, y)F.$$

Problem 3 **(home exercise)**

In the lectures we learned that the equilibrium equations

$$A^T CAx = f$$

are obtained from the principle of minimum potential energy. Show that an equivalent formulation is the principle of minimum complementary energy:

$$\min_y \frac{1}{2} y^T C^{-1} y \qquad \text{subject to } A^T y = f.$$

Hint: Look up the technique of Lagrange multipliers.

Problem 4

Read through section 4. Dynamical systems and eigenvalues from the handouts.

Problem 5

A project work for the next two weeks. Hints will be given in exercises at week 4 and this will be **home exercise for week 5.**

Make your own MATLAB program by which simple strusses can be analyzed. Make experiments on systems with and without unique solutions. In the latter case, check the nullspaces of the equilibrium equation and the force compatibility conditions. Compute and print the eigenmodes of some simple structures.

Problem 1

$N(B) = R(B^T)^\perp$.

$B : X \to Y$ lin. operator.

X, Y Banach spaces.

$N(B) = \{ x \in X \mid x \neq 0 \ , Bx = 0 \}$

$R(B^T) = \{ x \in X \mid \exists y \in Y \text{ s.t. } B^T y = x \}$

$R(B^T)^\perp = \{ x \in X \mid \exists y \in Y \text{ s.t. } B^T y = x$

and $\langle B^T y , z \rangle = 0 \ \forall z \in R(B^T)$

Take $x \in R(B^T)^\perp$.

$\Leftrightarrow \langle x, z \rangle_X = 0 \quad \forall z \in R(B^T)$

$\Leftrightarrow \langle x, B^T y \rangle_X = 0 \quad y$ is arbitrary

$\Leftrightarrow \langle Bx, y \rangle_Y = 0 \quad \forall y \in Y$.

$\Leftrightarrow Bx = 0$

$\Leftrightarrow x \in N(B)$.

Problem 2

Strain ε is uniform on $0 \le x \le y$ and $y < x \le L$. Moreover $\varepsilon = u'(x)$. Therefor $u(x)$ is linear on both intervals.

$$u(x) = \begin{cases} a_1 x + b_1 & 0 \le x < y \\ a_2 x + b_2 & y < x \le L \end{cases}$$

Boundary conditions $u(0) = u(L) = 0$ imply

$$u(x) = \begin{cases} a_1 x & 0 \le x < y \\ a_2 (x - L) & y < x \le L \end{cases}$$

We need two more conditions to deduce a_1 and a_2. These are compatibility conditions

1) $u(x)$ is continuous i.e. $u(y-) = u(y+)$.
2) force compatibility

$$-A\sigma(y-) - A\sigma(y+) + f = 0$$

Continuity: $u(y-) = u(y+)$

$$a_1 y = a_2 (y - L) \iff a_2 = \frac{y}{y-L} a_1$$

$$\Rightarrow \quad u(x) = \begin{cases} a_1 x & 0 \le x < y \\ a_1 \frac{y}{y-L} (x - L) & y < x \le L \end{cases}$$

Force compatibility:

$$\sigma = E\varepsilon = E u'(x) = \begin{cases} E a_1 & 0 \le x < y \\ E a_1 \dfrac{y}{y-L} & y < x \le L \end{cases}$$

$$\Rightarrow -A\,\sigma(y-) - A\,\sigma(y+) + f = 0$$

$$\Leftrightarrow \quad AEa_1\left(1 + \frac{y}{y-L}\right) = F$$

$$\Leftrightarrow \quad a_1 = \frac{y-L}{AE(2y-L)}\,F$$

$$\Rightarrow u(x) = \begin{cases} \dfrac{y-L}{AE(2y-L)}\,F x & 0 \le x < y \\ \dfrac{y}{AE(2y-L)}\,F(x-L) & y < x \le L \end{cases}$$

Define

$$K(x,y) := \begin{cases} \dfrac{y-L}{AE(2y-L)}\, x & 0 \le x \le y \\ \dfrac{y(x-L)}{AE(2y-L)} & y \le x \le L \end{cases}$$

$$\Rightarrow u(x) = K(x,y)\,F.$$

Problem 3

$$\min_y \tfrac{1}{2} y^T C^{-1} y \quad \text{subject to} \quad A^T y = f$$

With Lagrange multipliers, the solution is at an extremum of

$$L(y,\lambda) = \tfrac{1}{2} y^T C^{-1} y + \lambda^T [f - A^T y]$$

$$\nabla_\lambda L(y,\lambda) = f^T - y^T A = 0 \quad \text{(Nothing new...)}$$

$$\nabla_y L(y,\lambda) = \tfrac{1}{2}\left[(C^{-1}y)^T + y^T C^{-1}\right] - \lambda^T A^T = 0$$

$$\Leftrightarrow \quad y^T C^{-1} - \lambda^T A^T = 0$$

$$\Leftrightarrow \quad y^T = \lambda^T A^T C$$

Substitute this to $L(y,\lambda)$.

$$\Rightarrow \tfrac{1}{2} \lambda^T A^T \underbrace{C C^{-1}}_{=I} C A \lambda + \lambda^T f - \lambda^T A^T C A \lambda$$

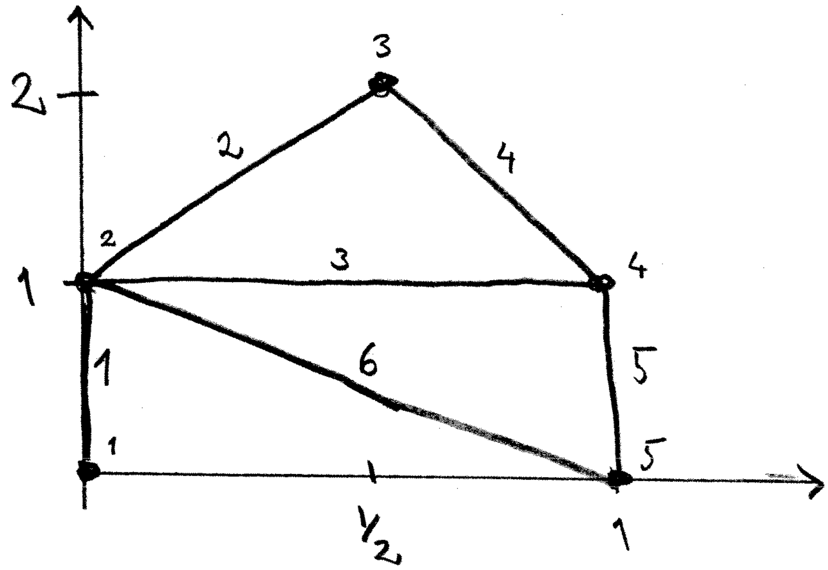
$$= -\tfrac{1}{2} \lambda^T \underbrace{A^T C A}_{\text{positive definite}} \lambda + \lambda^T f$$

$\Rightarrow$ extremum is

$$\min_\lambda \tfrac{1}{2} \lambda^T A^T C A \lambda - \lambda^T f.$$

Problem 5

Let us consider following truss



First we define joints

$$\text{joints} = \begin{bmatrix} 0 & 0 & 1/2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix} \begin{array}{l} \leftarrow \text{x-coord} \\ \leftarrow \text{y-coord} \end{array}$$

Then we define bars

$$\text{bars} = \begin{bmatrix} 1 & 2 & 2 & 3 & 4 & 2 \\ 2 & 3 & 4 & 4 & 5 & 5 \end{bmatrix} \begin{array}{l} \leftarrow \text{1st joint} \\ \leftarrow \text{2nd joint} \end{array}$$

Now you can use plot_truss.m to draw the stress.

Next you need to loop through joints and build matrix A. See page 6 in handouts for help. Function atan2 could prove to be useful.

To get eigenmodes, you can use function eig (or eigs).

Nullspaces are perhaps easiest to check with singular value decomposition i.e. with function svd. Matlab svd gives you matrixes U, S and V s.t.

$A = USV^T$, where S contains the singular values.