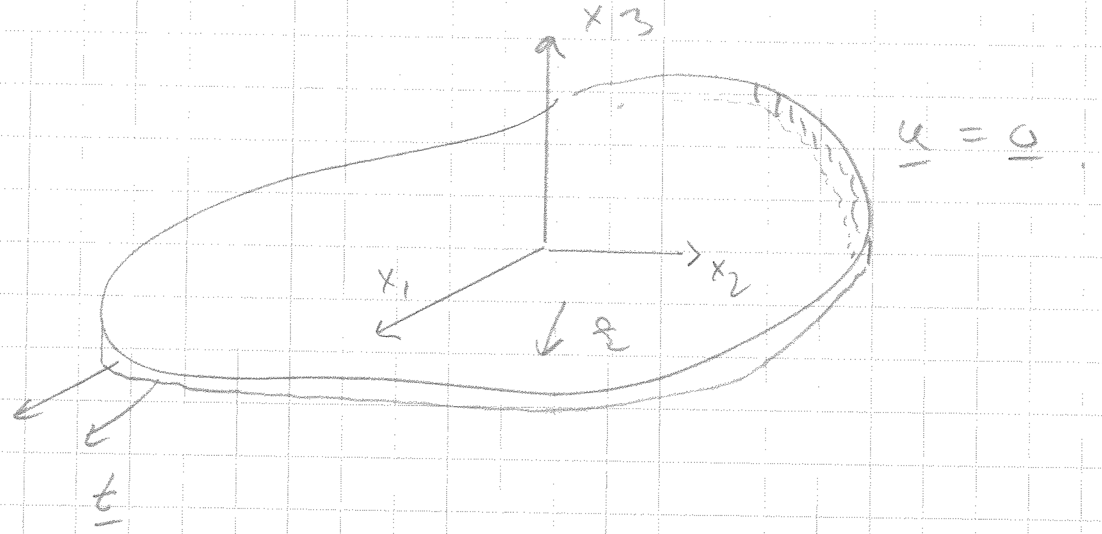
The plane stress and strain problem

Plane stress (Tasojännitys / Lerg)



Assumptions. The domain is thin in the x_3 - direction.

$$\Rightarrow \quad \sigma_{33} = \sigma_{23} = \sigma_{13} = 0.$$

and

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is advantageous to write Hooke's law in the form:

$$\underline{\underline{\varepsilon}} = \frac{1}{E}\left[(1+\nu)\,\underline{\underline{\sigma}} - \nu \operatorname{tr}(\underline{\underline{\sigma}})\,\underline{\underline{I}}\right]$$

We have

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu \sigma_{22}\right),$$

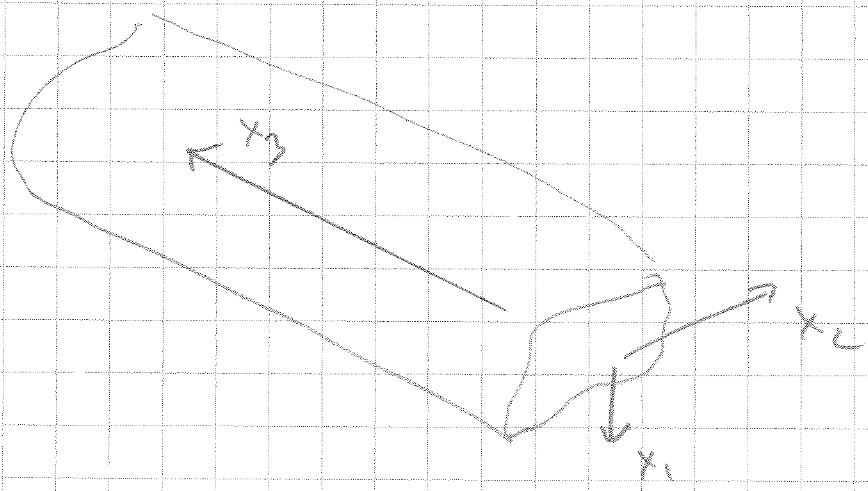
$$\varepsilon_{22} = \frac{1}{E}\left(\sigma_{22} - \nu \sigma_{11}\right),$$

$$\varepsilon_{13} = \frac{(1+\nu)}{E}\sigma_{13} = \frac{1}{2G}\sigma_{13},$$

and

$$\varepsilon_{33} = -\frac{\nu}{E}\left(\sigma_{11} + \sigma_{22}\right).$$

<u>The plain strain problem</u> (Taso venymä)



$$u_3 = 0 \quad \text{and}$$

$$u_1 = u_1(x_1, x_2), \quad u_2 = u_2(x_1, x_2).$$

This gives

$$\varepsilon_{3i} = 0, \quad i = 1, 2, 3.$$

Now, it is neater to use Hooke in the form

$$\underline{\underline{\sigma}} = 2\mu \underline{\underline{\varepsilon}} + \lambda \, \mathrm{tr}(\underline{\underline{\varepsilon}}) \underline{\underline{I}}$$

which gives

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda (\varepsilon_{11} + \varepsilon_{22}) \delta_{ij}, \quad i,j = 1,2,$$

and

$$\sigma_{33} = \lambda (\varepsilon_{11} + \varepsilon_{22}).$$

Using E and ν we have

$$\varepsilon_{11} = \frac{1-\nu^2}{E} \left(\sigma_{11} - \frac{\nu}{1-\nu} \sigma_{22} \right)$$

$$\varepsilon_{22} = \frac{1-\nu^2}{E} \left(\sigma_{22} - \frac{\nu}{1-\nu} \sigma_{11} \right)$$

and

$$\varepsilon_{12} = \frac{(1+\nu)}{E} \sigma_{12}.$$

The Airy stress function

For a plain stress problem with a vanishing body force the static equilibrium equations are

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0,$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0.$$

The classical way to search for solutions is to use the Airy stress function $\Phi(x_1, x_2)$ satisfying

$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial x_2^2}, \quad \sigma_{22} = \frac{\partial^2 \Phi}{\partial x_1^2}$$

$$\sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}.$$

This satisfies the equilibrium equations:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = \frac{\partial^2 \Phi}{\partial x_1 \partial x_2^2} + \frac{\partial}{\partial x_2}\left(-\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}\right) = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = \frac{\partial}{\partial x_1}\left(-\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial^2 \Phi}{\partial x_1^2}\right) = 0$$

Theorem. The Airy function satisfies the biharmonic equation

$$\Delta^2 \Phi = 0.$$

Proof. First, we note that the strains satisfy the compatibility condition

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^{\,2}} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^{\,2}} = 2\,\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}.$$

Proof: The left hand side is

$$\varepsilon_{11,22} + \varepsilon_{22,11} = u_{1,122} + u_{2,211}.$$

The right hand side:

$$2\,\varepsilon_{12,12} = u_{1,212} + u_{2,112}. \qquad \text{O.K.!}$$

Next, we substitute the relations

$$\varepsilon_{11} = \frac{1}{E}\left(\Phi_{,22} - \nu\,\Phi_{,11}\right),$$

$$\varepsilon_{22} = \frac{1}{E}\left(\Phi_{,11} - \nu\,\Phi_{,22}\right),$$

$$\varepsilon_{12} = -\frac{(1+\nu)}{E}\,\Phi_{,12},$$

which gives

$$\varepsilon_{11,22} + \varepsilon_{22,11} - 2\,\varepsilon_{12,12}$$

$$= \frac{1}{E}\left(\Phi_{,2222} - \nu\,\Phi_{,1122}\right)$$

$$+ \frac{1}{E}\left(\Phi_{,1111} - \nu\,\Phi_{,2211}\right)$$

$$+ \frac{2(1+\nu)}{E}\,\Phi_{,1212}$$

$$= \frac{1}{E}\left(\Phi_{,1111} + 2\,\Phi_{,1122} + \Phi_{,2222}\right).$$

$$= \frac{1}{E}\,\Delta^2\Phi = 0.$$

$$\left(\Delta^2 = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)^2 = \frac{\partial^4}{\partial x_1^4} + 2\,\frac{\partial^4}{\partial x_1^2\,\partial x_2^2} + \frac{\partial^4}{\partial x_2^4}\right).$$

$\square$.

The use of Airy today:

1) Theoretical considerations (e.g. to find corner singularities).
2) "Benchmark" computations.

2. Calculus of Variations

2.1. One dimensional problems

Consider the interval $[a,b] \subset \mathbb{R}$.

The problem is to find the function

$u(x) = u : [a,b] \to \mathbb{R}$ such that

the variational integral

$$I(u) = \int_a^b F(x, u(x), u'(x))\, dx$$

attains an extremal value.

The unknow u is sought from

a space of sufficiently regular

functions, which all satisfy

the boundary conditions

$$u(a) = 0 \quad , u(b) = 0.$$

In order to derive a

necessary condition for the extremum

we choose a function $\eta(x)$

such that $\eta(a) = 0$, $\eta(b) = 0$.

We the study $I(u + \varepsilon\eta)$,

where $\varepsilon \in \mathbb{R}$.

We obtain a real valued function

$$f(\varepsilon) = I(u + \varepsilon\eta)$$

The condition for u to give an extremum is now that

$$f'(0) = 0.$$

We compute

$$f'(\varepsilon) = \frac{d}{d\varepsilon} \int_a^b F(x, u(x) + \varepsilon\eta(x), u'(x) + \varepsilon\eta'(x))\, dx$$

$$= \int_a^b \left[\frac{\partial}{\partial u} F(x, u + \varepsilon\eta, u' + \varepsilon\eta')\,\eta + \frac{\partial}{\partial u'} F(x, u + \varepsilon\eta, u' + \varepsilon\eta')\,\eta' \right] dx$$

The second term above is treated by integration by parts and using the conditions $\eta(a) = 0$, $\eta(b) = 0$:

$$\int_a^b \frac{\partial}{\partial u'} F(x, u(x) + \varepsilon\eta(x), u'(x) + \varepsilon\eta'(x))\,\eta'(x)\, dx$$

$$= \Big|_a^b\, \eta(x) \frac{\partial}{\partial u'} F(x, u + \varepsilon\eta, u' + \varepsilon\eta')$$

$$- \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial u'}(x, u + \varepsilon\eta, u' + \varepsilon\eta') \right) \eta\, dx$$

$$= -\int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial u'}(x, u+\varepsilon\eta, u'+\varepsilon\eta')\right)\eta \, dx.$$

Hence, we have

$$f'(0) = \int_a^b \left[\frac{\partial F}{\partial u}(x, u(x), u'(x)) - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}(x, u(x), u'(x))\right)\right]\eta(x)\, dx$$

$$= 0.$$

Now, η is arbitrary and hence it must hold

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left[\frac{\partial F}{\partial u'}\right] = 0.$$

This is the Euler (or E-Lagrange) equation for u.

<u>Example</u>. Find the curve $y = u(x)$ of least length that connects the points (x_1, y_1) and (x_2, y_2).

The length of the curve is

$$L(u) = \int_{x_1}^{x_2} (1 + u'(x)^2)^{1/2}\, dx.$$

We have

$$F(x, u, u') = (1+(u')^2)^{1/2}$$

and hence

$$\frac{\partial F}{\partial u} = 0 \ , \quad \frac{\partial F}{\partial u'} = u'(1+(u')^2)^{-1/2} .$$

The Euler-Lagrange differential equation is then

$$\frac{d}{dx}\left(u'(x)\,(1+u'(x)^2)^{-1/2} \right) = 0$$

which gives

$$u'(x)\,(1+u'(x)^2)^{-1/2} = c$$

and by solving for $u'(x)$:

$$u'(x) = \pm \frac{c}{(1-c^2)^{1/2}} = \text{a constant}.$$

Hence we see that $u'(x)$ is a constant and the solution is a line.

$\square$.

The isoparametric problem

is: Find $u = u(x)$, $u(a) = u(b) = 0$ such that

$$I(u) = \int_a^b F(x, u, u')\, dx$$

subject to the side condition

$$J(u) = \int_a^b G(x, u, u')\, dx = C \quad (\text{const.})$$

For seeking an extremum, we have to assume that u does not yield an extremum for $J(u)$, i.e. it holds

$$\frac{\partial G}{\partial u} - \frac{d}{dx}\left(\frac{\partial G}{\partial u'}\right) \not\equiv 0.$$

Next, we choose two "test functions" η_1, η_2, both satisfying

$$\eta_i(a) = \eta_i(b) = 0.$$

We consider the variation

$$u \leftarrow u + \varepsilon_1 \eta_1 + \varepsilon_2 \eta_2 .$$

Let

$$f(\varepsilon_1, \varepsilon_2) = I(u + \varepsilon_1 \eta_1 + \varepsilon_2 \eta_2)$$

and

$$g(\varepsilon_1, \varepsilon_2) = J(u + \varepsilon_1 \eta_1 + \varepsilon_2 \eta_2) - C.$$

We obtain a constrained extremal problem; $\varepsilon_1 = \varepsilon_2 = 0$ should be the solution to

$$f(\varepsilon_1, \varepsilon_2) \to \text{Ext.}$$

subject to

$$g(\varepsilon_1, \varepsilon_2) = 0.$$

For this we use the method of Lagrange multipliers. Let

$$L(\varepsilon_1, \varepsilon_2, \lambda) = f(\varepsilon_1, \varepsilon_2) - \lambda g(\varepsilon_1, \varepsilon_2).$$

The optimality condition is

$$\left.\frac{\partial L}{\partial \varepsilon_i}\right|_{\substack{\varepsilon_1 = 0 \\ \varepsilon_2 = 0}} = 0.$$

As before (differentiating inside the integral, and integrating by parts, using $\eta_i(a) = \eta_i(b)$) we get

$$\int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) - \lambda\left(\frac{\partial G}{\partial u} - \frac{d}{dx}\left(\frac{\partial G}{\partial u'}\right)\right)\right] \eta_i \, dx = 0.$$

We have not yet said anything about η_1 and η_2. Since we assumed that

$$\frac{\partial G}{\partial u} - \frac{d}{dx}\left(\frac{\partial G}{\partial u'}\right) \not\equiv 0$$

There exist η_2 such that

$$\int_a^b \left[\frac{\partial G}{\partial u} - \frac{d}{dx}\left(\frac{\partial G}{\partial u'}\right)\right] \eta_2 \, dx \neq 0.$$

Hence, by fixing η_2 in this way we can uniquely solve λ:

$$\lambda = \frac{\int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right] \eta_2 \, dx}{\int_a^b \left[\frac{\partial G}{\partial u} - \frac{d}{dx}\left(\frac{\partial G}{\partial u'}\right)\right] \eta_2 \, dx}.$$

This λ does not depend on η_1!

Hence, we are free to vary η_1 giving

$$\int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) - \lambda\left(\frac{\partial G}{\partial u} - \frac{d}{dx}\left(\frac{\partial G}{\partial u'}\right)\right)\right] \eta_1 \, dx = 0.$$

From this we get the Euler-Lagrange equation

$$\frac{\partial F^*}{\partial u} - \frac{d}{dx}\left(\frac{\partial F^*}{\partial u'}\right) = 0,$$

for

$$F^*(x,u,u') = F(x,u,u') - \lambda G(x,u,u').$$

$\square$.