Helsinki University of Technology
Mat-5.156 Theory of Elasticity **Stenberg / Tuominen**
2. exercise on Tuesday 15th February 2005

1. Calculate the maximum value for shear force and corresponding direction in case, when the stress tensor is given in coordinates of principal stresses and $\sigma_2 = \sigma_3$:

$$\underline{\underline{\tau}} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_2 \end{pmatrix}.$$

2. The definition of inner product for tensors $\underline{\underline{s}} = \{s_{ij}\}$ and $\underline{\underline{t}} = \{t_{ij}\}$ is

$$\underline{\underline{s}} : \underline{\underline{t}} = s_{ij}t_{ij}$$

(Einstein's summing notation). Show that

(a) if $\underline{\underline{s}}$ is symmetric ($\underline{\underline{s}}^T = \underline{\underline{s}}$), then

$$\underline{\underline{s}} : \underline{\underline{t}} = \underline{\underline{s}} : \underline{\underline{t}}^T = \underline{\underline{s}} : \{\frac{1}{2}(\underline{\underline{t}} + \underline{\underline{t}}^T)\}.$$

(b) if $\underline{\underline{\omega}}$ is antisymmetric ($\underline{\underline{\omega}}^T = -\underline{\underline{\omega}}$), then

$$\underline{\underline{\omega}} : \underline{\underline{t}} = -\underline{\underline{\omega}} : \underline{\underline{t}}^T = \underline{\underline{\omega}} : \{\frac{1}{2}(\underline{\underline{t}} - \underline{\underline{t}}^T)\}.$$

(c) if $\underline{\underline{t}} : \underline{\underline{s}} = 0$ for every symmetric $\underline{\underline{s}}$, then $\underline{\underline{t}}$ is antisymmetric.

(d) if $\underline{\underline{t}} : \underline{\underline{\omega}} = 0$ for every antisymmetric $\underline{\underline{\omega}}$, then $\underline{\underline{t}}$ is symmetric.

(e) if $\underline{\underline{s}}$ is symmetric and $\underline{\underline{\omega}}$ is antisymmetric, then $\underline{\underline{s}} : \underline{\underline{\omega}} = 0$.

(f) every tensor $\underline{\underline{t}}$ can be written uniquely as a sum of symmetric and antisymmetric tensors.

3. Show that both large and small strains

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}),$$
$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

can be transformed with orthogonal transformation of coordinates ($\epsilon'_{kl} = a_{ki}a_{lj}\epsilon_{ij}$).

4. (Home exercise.) Proof that $\varepsilon_{kk} = \sum_{k=1}^{3} \varepsilon_{kk} = \mathrm{tr}(\underline{\underline{\varepsilon}})$ represents physically the proportional change of volume.
Assume that there is the state of hydrostatic pressure in a body:

$$\underline{\underline{\tau}} = -p\,\underline{\underline{I}}, \quad p = \text{vakio} > 0.$$

Show that

$$\varepsilon_{kk} = -\frac{1}{\kappa}p,$$

where κ is bulk modulus

$$\kappa = \lambda + \frac{2}{3}\mu.$$

Condition $\kappa > 0$ is natural, isn't it? Write the expression for κ in terms of Young's modulus E and Poisson ratio ν. Which condition arises for ν?

5. For arbitrary stress tensor $\underline{\underline{\tau}}$ so-called hydrostatic part is defined as

$$\frac{\tau_{kk}}{3} = \tau_m,$$

and stress deviator is

$$\overline{\tau}_{ij} = \tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij}.$$

Similarly the hydrostatic part for the strain is defined as

$$\frac{\varepsilon_{kk}}{3} = \varepsilon_m,$$

and deformation deviator is

$$\overline{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}.$$

Show that the elastic energy density can be written in form

$$U_0 = U_{vol} + U_{def},$$

where

$$U_{vol} = \frac{1}{6}\tau_{kk}\varepsilon_{ll}$$

is energy density related to change in volume, and

$$U_{def} = \frac{1}{2}\overline{\tau}_{ij}\,\overline{\varepsilon}_{ij}$$

is energy density related to deformation.
Show also that for isotropic material holds true that

$$U_{vol} = \frac{\kappa}{2}\varepsilon_{kk}\varepsilon_{ll}$$

and

$$U_{def} = G\overline{\varepsilon}_{ij}\,\overline{\varepsilon}_{ij}.$$