Helsinki University of Technology
Mathematics
Somersalo/Bingham

Mat-1.198 Scattering Theory

8$^{\text{th}}$ set of exercises, 2.4.2003

1. Consider the one-dimensional potential scattering problem: Let $q : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous potential, $q(x) = 0$ for $|x| > M$, and u a solution of the equation

   $$u'' + k^2 u = qu.$$

   Write

   $$u(x) = u_{\text{inc}}(x) + u_{\text{sc}}(x),$$

   where the incoming field is a left-moving plane wave sent from $+\infty$,

   $$u_{\text{inc}}(x) = e^{-ikx}.$$

   Part of this wave is reflected back, part is transmitted through, so that

   $$u(x) = \begin{cases} Re^{ikx} + e^{-ikx}, & x > M \\ Te^{-ikx}, & x < -M. \end{cases}$$

   (a) Write the appropriate radiation condition in this case.

   (b) Derive the Lippmann-Schwinger integral equation.

   (c) Find the integral representations of the reflection and transmission coefficients R and T.

   (d) What is the Born approximations for R and T?

2. Prove that the scattering is unitary:

   $$|R|^2 + |T|^2 = 1.$$

   (Hint: Consider the Wronskian of the solutions $u(x)$ and $\overline{u(x)}$.)

3. Prove the inversion formula for Abel's integral equation: If

   $$h(y) = \int_0^y \frac{f(t)}{(y-t)^\alpha} dt, \quad 0 < \alpha < 1,$$

   then

   $$f(t) = \frac{\sin \pi\alpha}{\pi} \frac{d}{dt} \int_0^t \frac{h(y)}{(t-y)^{1-\alpha}} dy.$$

   (Hint: Use Laplace transform and the convolution theorem.)