Helsinki University of Technology
Mathematics

Somersalo/Bingham

Mat-1.198 Scattering Theory

1st set of exercises, 29.1.2003

By doing the exercises, you can earn points that will be taken into account when determining your grade for the course: Do these exercises in advance, mark them as solved on the list at the exercise session and be prepared to present your solution on the blackboard.

1. Show that in the two dimensional case, the fundamental solution of the Helmholtz equation is
$$\Phi(x) = \frac{i}{4} H_0^{(1)}(k|x|),$$
i.e., verify the multiplicative constant. Further, find the explicit form of the asymptotic expansion
$$\Phi(x-y) \sim \frac{e^{ik|x|}}{\sqrt{|x|}} \left[f(\hat{x}, y) + O\left(\frac{1}{|x|}\right)\right],$$
when $|x| \to \infty$ and $|y| < R$.

2. Harmonic polynomials in $\mathbb{R}^2$: A polynomial $P_n(x,y)$ of degree $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ in $\mathbb{R}^2$ is *homogenous*, if
$$P_n(\lambda x, \lambda y) = \lambda^n P_n(x,y) \quad \text{for all } \lambda > 0,\ (x,y) \in \mathbb{R}^2$$
and *harmonic*, if
$$\Delta P_n(x,y) = 0.$$
The two-dimensional equivalent to spherical harmonics of $\mathbb{R}^3$ are defined as the restrictions of the homogenous harmonic polynomials to the unit circle S^1, or simply
$$Y_n(\theta) = P_n(\cos\theta, \sin\theta), \quad 0 \le \theta < 2\pi.$$
Show that a basis for these functions is given by $\{e^{\pm in\theta}\}$.

3. Verify the recurrence relation
$$J_{p-1}(z) - J_{p+1}(z) = 2J_p'(z).$$
In particular, deduce the useful formula
$$J_0'(z) = -J_1(z).$$
(Note: This formula holds for the Neumann function N_p as well, and thus also for the Hankel functions $H_p^{(j)} = J_p + (-1)^{j-1} N_p$, $j \in \{1, 2\}$.)

4. The *Hankel transform* $\mathcal{H}_p$ is defined for functions $f : \mathbb{R}_+ \to \mathbb{C}$ as

$$\mathcal{H}_p(f)(k) = \int_0^\infty J_p(kr) f(r) r dr, \quad p = 0, 1, \ldots$$

whenever this integral is convergent. For functions $f : \mathbb{R}^2 \to \mathbb{C}$, we define the Fourier transform as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-i\xi\cdot x} f(x) dx, \quad \xi \in \mathbb{R}^2.$$

Find an expression for the two-dimensional Fourier transform in terms of the Hankel transforms and the trigonometric Fourier series representation

$$f(x) = \sum_{m=-\infty}^{\infty} f_m(r) e^{im\theta}, \qquad x = (r\cos\theta, r\sin\theta).$$

(Assume here that all the necessary integrals and series are convergent, i.e., a formal derivation suffices.)

Useful formulae which you may assume when doing the excercises:

$$J_p(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(z\sin\theta - p\theta)} d\theta = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(p+j)!} \left(\frac{z}{2}\right)^{p+2j}$$

The series expansion also holds for $p \in \mathbb{Z}_-$ with the following interpretation for $j \in \mathbb{Z}_-$:

$$\frac{1}{j!} = \frac{j+1}{(j+1)!} = \frac{(j+1)(j+2)}{(j+2)!} = \cdots = \frac{(j+1)(j+2)\cdots(-1)0}{0!} = \frac{0}{1} = 0.$$

$$N_0(z) = \frac{2}{\pi}\left(\ln\frac{z}{2} + \gamma\right) + O(z^2), \qquad N_p(z) = \left(\frac{z}{2}\right)^{-p}\left[-\frac{(p-1)!}{\pi} + o(1)\right], \qquad z \to 0, \operatorname{Re} z > 0$$

$$H_p^{(j)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{i(-1)^j[(2p+1)\frac{\pi}{4} - z]}\left[1 + O\left(\frac{1}{|z|}\right)\right], \qquad |z| \to \infty, -\pi < \arg z < 2\pi, j \in \{1,2\}$$

The notation Y_p is also often used for the Neumann function N_p in literature, for instance in the book by Colton and Kress.