Multiresolutions

1. Definitions and basic properties

In Chapter 1 we gave an example of how one starting from one function φ can construct a scale of subspaces $\ldots \subset V_m \subset V_{m-1} \subset \ldots$. This is what we mean by a multiresolution (or multiresolution analysis) of $L^2(\mathbb{R};\mathbb{C})$. A more precise definition is the following.

Definition 4.1. *$(\{V_m\}_{m\in\mathbb{Z}}, \varphi)$ is an orthonormal multiresolution of $L^2(\mathbb{R};\mathbb{C})$ provided that*

(4.1) *$\varphi \in L^2(\mathbb{R};\mathbb{C})$ and V_m is, for each $m \in \mathbb{Z}$, the closed subspace of $L^2(\mathbb{R};\mathbb{C})$ spanned by $\{\varphi(2^{-m}\bullet -k)\}_{k\in\mathbb{Z}}$,*

(4.2) *$V_m \subset V_{m-1}$, $m \in \mathbb{Z}$,*

(4.3) *$\lim_{m\to-\infty} V_m = L^2(\mathbb{R};\mathbb{C})$, i.e. $\lim_{m\to-\infty} \inf_{g\in V_m} \|f-g\|_{L^2(\mathbb{R})} = 0$ for every $f \in L^2(\mathbb{R};\mathbb{C})$,*

(4.4) *$\{\varphi(\bullet - k)\}_{k\in\mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{R};\mathbb{C})$ and thus an orthonormal basis for the closed subspace it spans.*

The function φ is said to be a father wavelet or scaling function, and is said to generate the multiresolution.

The definition of a multiresolution is often given in a slightly different form, so that the fact that V_m is spanned by $\{\varphi(2^{-m}\bullet -k)\}_{k\in\mathbb{Z}}$ is a consequence of the other conditions. The main motivation for this formulation is that it makes it easier to see which properties of the function φ and the filter to be defined below, depend on which properties of the multiresolution.

Lemma 4.2. *Let $g \in L^2(\mathbb{R};\mathbb{C})$ and let $g_k = g(\bullet - k)$. Then $(g_k)_{k\in\mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R};\mathbb{C})$ if and only if*

$$\sum_{k\in\mathbb{Z}} \left|\hat{g}(\bullet + k)\right|^2 \overset{a.e.}{=} 1.$$

Proof. Since g belongs to $L^2(\mathbb{R};\mathbb{C})$, it follows from Theorem 2.4 that its Fourier transform $\hat{g}$ belongs to $L^2(\mathbb{R};\mathbb{C})$ as well, and this means in particular that the function

$$h \overset{\text{def}}{=} \sum_{k\in\mathbb{Z}} |\hat{g}(\bullet + k)|^2 \tag{4.5}$$

belongs to $L^1([0,1];\mathbb{R})$. By the uniqueness of the Fourier coefficients of a periodic function it is clear that $h = 1$ a.e. in $[0,1]$ (or equivalently on $\mathbb{R}$) if and only if

$$\hat{h}(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases} \tag{4.6}$$

A calculation where we use the definition of h, Plancherel's theorem, and the fact that the Fourier transform of $g(\bullet - k)$ is $\mathrm{e}^{-\mathrm{i}2\pi k\bullet}\hat{g}$, gives

$$\begin{aligned} \hat{h}(k) &= \int_0^1 \mathrm{e}^{-\mathrm{i}2\pi k\omega} h(\omega)\,\mathrm{d}\omega = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi k\omega} \hat{g}(\omega)\overline{\hat{g}(\omega)}\,\mathrm{d}\omega \\ &= \int_{\mathbb{R}} g(x-k)\overline{g(x)}\,\mathrm{d}x = \int_{\mathbb{R}} g(x-k-m)\overline{g(x-m)}\,\mathrm{d}x = \langle g_{k+m}, g_m, \cdot\rangle \end{aligned} \tag{4.7}$$

Thus we see that the orthonormality of the set $\{g_k\}_{k\in\mathbb{Z}}$ is equivalent to (4.6) and this is what we had to prove. □

Proposition 4.3. *Let* (4.1) *and* (4.4) *hold. Then $\left\{2^{-\frac{m}{2}}\varphi(2^{-m}\bullet - k)\right\}_{k\in\mathbb{Z}}$, is an orthonormal basis for V_m for each $m \in \mathbb{Z}$ and $\bigcap_{m\in\mathbb{Z}} V_m = \{0\}$.*

The proof is left as an exercise.

Now we can start to approach the heart of the matter in the following result.

Theorem 4.4. *Let* (4.1), (4.2), *and* (4.4) *hold. Then the sequence α, called the filter associated with φ and defined by*

$$\alpha(k) = \langle \varphi, \varphi(2\bullet - k)\rangle, \quad k \in \mathbb{Z} \tag{4.8}$$

satisfies $\sum_{k\in\mathbb{Z}}|\alpha(k)|^2 = \frac{1}{2}$,

$$\varphi = 2\sum_{k\in\mathbb{Z}}\alpha(k)\varphi(2\bullet - k), \tag{4.9}$$

or, equivalently,

$$\hat{\varphi}(2\bullet) = \hat{\alpha}(\bullet)\hat{\varphi}(\bullet), \tag{4.10}$$

and

$$|\hat{\alpha}(\bullet)|^2 + \left|\hat{\alpha}\left(\bullet + \tfrac{1}{2}\right)\right|^2 = 1 \quad \textit{a.e. on} \quad \mathbb{R}. \tag{4.11}$$

A summable sequence $\{\alpha_j\}_{j\in\mathbb{Z}}$ satisfying (4.11) above, is said to be a conjugate quadrature filter.

Proof. Since $\varphi \in V_0 \subset V_{-1}$ and $\left\{\sqrt{2}\varphi(2\bullet - k)\right\}_{k\in\mathbb{Z}}$ is a basis for V_{-1} (see proposition 4.3), it follows that

$$\varphi = \sum_{k\in\mathbb{Z}}\left\langle \varphi, \sqrt{2}\varphi(2\bullet - k)\right\rangle \sqrt{2}\varphi(2\bullet - k). \tag{4.12}$$

Thus we see that (4.9) follows from (4.8) and the fact that $\sum_{k\in\mathbb{Z}}|\alpha(k)|^2 = \frac{1}{2}$ is a consequence of Theorem 3.4.(iii).

Taking the Fourier transform of both sides of (4.9) we get

$$\hat{\varphi}(\omega) = \sum_{k\in\mathbb{Z}}\alpha(k)\mathrm{e}^{-\mathrm{i}2\pi k\omega/2}\hat{\varphi}\left(\tfrac{\omega}{2}\right), \tag{4.13}$$

which is the same as (4.10).

By lemma 4.2, the orthonormality of $\{\varphi(\bullet - k)\}_{k\in\mathbb{Z}}$, and (4.10) it follows that

$$\begin{aligned} 1 &= \sum_{k\in\mathbb{Z}}|\hat{\varphi}(2\omega + k)|^2 \\ &= \sum_{m\in\mathbb{Z}}\left|\hat{\alpha}\left(\omega + \tfrac{2m}{2}\right)\right|^2\left|\hat{\varphi}\left(\omega + \tfrac{2m}{2}\right)\right|^2 + \sum_{m\in\mathbb{Z}}\left|\hat{\alpha}\left(\omega + \tfrac{2m+1}{2}\right)\right|^2\left|\hat{\varphi}\left(\omega + \tfrac{2m+1}{2}\right)\right|^2 \\ &= |\hat{\alpha}(\omega)|^2\sum_{m\in\mathbb{Z}}|\hat{\varphi}(\omega + m)|^2 + \left|\hat{\alpha}\left(\omega + \tfrac{1}{2}\right)\right|^2\sum_{m\in\mathbb{Z}}\left|\hat{\varphi}\left(\omega + \tfrac{1}{2} + m\right)\right|^2 \\ &= |\hat{\alpha}(\omega)|^2 + \left|\hat{\alpha}\left(\omega + \tfrac{1}{2}\right)\right|^2. \end{aligned} \tag{4.14}$$

This completes the proof. □

In order to get a basis for the orthogonal complement of V_m in V_{m-1} we consider a general result on the "splitting" of an orthonormal basis.

Theorem 4.5. *Suppose that* $(e_k)_{k\in\mathbb{Z}}$ *is an orthonormal basis in a Hilbert space* H, *let* α *and* $\beta \in \ell^2(\mathbb{Z})$, *and define sequences* $(u_k)_{k\in\mathbb{Z}}$ *and* $(v_k)_{k\in\mathbb{Z}}$ *in* H *as follows:*

$$u_{\underline{k}} = \sqrt{2}(\alpha * e)(2\underline{k}), \tag{4.15}$$

$$v_{\underline{k}} = \sqrt{2}(\beta * e)(2\underline{k}). \tag{4.16}$$

Let U *and* V *denote the closed subspaces of* H *spanned by the sequences* $(u_k)_{k\in\mathbb{Z}}$ *and* $(v_k)_{k\in\mathbb{Z}}$.

Then $(u_k)_{k\in\mathbb{Z}}$ *is an orthonormal basis of* U, $(v_k)_{k\in\mathbb{Z}}$ *is an orthonormal basis of* V, $U \perp V$, *and* $U \oplus V = H$ *if and only if the matrix*

$$\begin{pmatrix} \hat{\alpha}(\frac{\omega}{2}) & \hat{\alpha}(\frac{\omega}{2}+\frac{1}{2}) \\ \hat{\beta}(\frac{\omega}{2}) & \hat{\beta}(\frac{\omega}{2}+\frac{1}{2}) \end{pmatrix} \quad \textit{is unitary for almost every } \omega \in \mathbb{R}. \tag{4.17}$$

The convolution notation used above means that for example $u_k = \sqrt{2}\sum_{j\in\mathbb{Z}} \alpha(j)e_{2k-j}$ for all $k \in \mathbb{Z}$.

Proof. Since $(e_k)_{k\in\mathbb{Z}}$ is an orthonormal sequence, some straightforward calculations show that $(u_k)_{k\in\mathbb{Z}}$ and $(v_k)_{k\in\mathbb{Z}}$ are orthonormal sequences and $U \perp V$ if and only if

$$\begin{aligned} 2\sum_{j\in Z} \alpha(j)\overline{\alpha(j+2m-2n)} &= \delta_{m,n}, \\ 2\sum_{j\in Z} \beta(j)\overline{\beta(j+2m-2n)} &= \delta_{m,n}, \\ 2\sum_{j\in Z} \alpha(j)\overline{\beta(j+2m-2n)} &= 0, \end{aligned} \tag{4.18}$$

for all m and $n \in \mathbb{Z}$, (where we have $\delta_{m,m} = 1$ and $\delta_{m,n} = 0$ if $m \neq n$). In order to treat all these different cases at the same time we let a and $b \in \ell^2(\mathbb{Z})$ and define

$$c(m) = 2\sum_{j\in\mathbb{Z}} a(j)\overline{b(j+2m)}, \quad m \in \mathbb{Z}.$$

Since $\sum_{j\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}j} b(j+2m) = \mathrm{e}^{\mathrm{i}2\pi 2\underline{\omega}m}\hat{b}(\underline{\omega})$ if follows from Plancherel's theorem, (Theorem 2.4.(c)), that

$$\begin{aligned} c(m) = 2\int_0^1 \mathrm{e}^{-\mathrm{i}2\pi 2\omega m}\hat{a}(\omega)\overline{\hat{b}(\omega)}\,\mathrm{d}\omega &= 2\int_0^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}2\pi 2\omega m}\hat{a}(\omega)\overline{\hat{b}(\omega)}\,\mathrm{d}\omega \\ &\quad + 2\int_{\frac{1}{2}}^1 \mathrm{e}^{-\mathrm{i}2\pi 2\omega m}\hat{a}(\omega)\overline{\hat{b}(\omega)}\,\mathrm{d}\omega \\ &= 2\int_0^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}2\pi 2\omega m}\left(\hat{a}(\omega)\overline{\hat{b}(\omega)} + \hat{a}(\omega+\tfrac{1}{2})\overline{\hat{b}(\omega\tfrac{1}{2})}\right)\,\mathrm{d}\omega \\ &= \int_0^1 \mathrm{e}^{-\mathrm{i}2\pi\omega m}\left(\hat{a}(\tfrac{\omega}{2})\overline{\hat{b}(\tfrac{\omega}{2})} + \hat{a}(\tfrac{\omega}{2}+\tfrac{1}{2})\overline{\hat{b}(\tfrac{\omega}{2}+\tfrac{1}{2})}\right)\,\mathrm{d}\omega. \end{aligned}$$

From the uniqueness of the Fourier transform we seet that $c(m) = 0$ for all m if and only if $\hat{a}(\frac{1}{2}\underline{\omega})\overline{\hat{b}(\frac{1}{2}\underline{\omega})} + \hat{a}(\frac{1}{2}\underline{\omega}+\frac{1}{2})\overline{\hat{b}(\frac{1}{2}\underline{\omega}+\frac{1}{2})} \overset{\text{a.e.}}{=} 0$ and $c_{\underline{m}} = \delta_{0,\underline{m}}$ if and only if $\hat{a}(\frac{1}{2}\underline{\omega})\overline{\hat{b}(\frac{1}{2}\underline{\omega})} + \hat{a}(\frac{1}{2}\underline{\omega}+\frac{1}{2})\overline{\hat{b}(\frac{1}{2}\underline{\omega}+\frac{1}{2})} \overset{\text{a.e.}}{=} 1$. When we apply these results to the expressions in (4.18) we conclude that $(u_k)_{k\in\mathbb{Z}}$ and $(v_k)_{k\in\mathbb{Z}}$ are orthonormal sequences and $U \perp V$ if and only if the matrix given in the statement of the theorem is unitary.

It remains to show that when these conditions hold, then it is also true that $U \oplus V = H$. Suppose that this is not the case, but that there is an element $f \in H \setminus (U \oplus V)$. Thus we may assume that $f \neq 0$ but $f \perp (U \oplus V)$ so that $f \perp U$ and $f \perp V$. Thus it follows that $\langle u_n, f\rangle = \langle v_n, f\rangle = 0$ for all n and if we let $\gamma(j) = \langle f, e_{-j}\rangle$, then it follows that

$$\begin{aligned} 0 = \langle u_{-m}, f\rangle &= \sqrt{2}\sum_{j\in\mathbb{Z}} \alpha(j)\overline{\gamma(j+2m)}, \\ 0 = \langle v_{-m}, f\rangle &= \sqrt{2}\sum_{j\in\mathbb{Z}} \beta(j)\overline{\gamma(j+2m)}, \end{aligned}$$

for all $m \in \mathbb{Z}$. By the same argument that was used above, we conclude that this implies that

$$\begin{pmatrix} \hat{\alpha}(\frac{1}{2}\underline{\omega}) & \hat{\alpha}(\frac{1}{2}\underline{\omega}+\frac{1}{2}) \\ \hat{\beta}(\frac{1}{2}\underline{\omega}) & \hat{\beta}(\frac{1}{2}\underline{\omega}+\frac{1}{2}) \end{pmatrix} \begin{pmatrix} \overline{\hat{\gamma}(\frac{1}{2}\underline{\omega})} \\ \overline{\hat{\gamma}(\frac{1}{2}\underline{\omega}+\frac{1}{2})} \end{pmatrix} \overset{\text{a.e.}}{=} 0,$$

on $[0,1]$ But since this matrix is unitary, hence invertible, it follows that $\hat{\gamma}(\frac{\underline{\omega}}{2}) = \overset{\text{a.e.}}{=} \hat{\gamma}(\frac{\underline{\omega}}{2}+\frac{1}{2}) \overset{\text{a.e.}}{=} 0$ on $[0,1]$ and therefore $\hat{\gamma}(\underline{\omega}) \overset{\text{a.e.}}{=} 0$ on $[0,1]$ and by the uniqueness of the Fourier transform we see that $\langle f, e_k\rangle = 0$ for all k and hence $f = 0$. This completes the proof. □

Definition 4.6. *Let (4.1), (4.2), and (4.4) hold and let α be the filter associated with φ. Then the mother wavelet associated with φ is the function*

$$\psi = 2\sum_{k\in\mathbb{Z}}(-1)^k\overline{\alpha(1-k)}\varphi(2\bullet-k). \tag{4.19}$$

Theorem 4.7. *Let (4.1), (4.2), and (4.4) hold, and let ψ be the associated mother wavelet. Then $\{2^{-\frac{m}{2}}\psi(2^{-m}\bullet-k)\}_{k\in\mathbb{Z}}$ is for each $m\in\mathbb{Z}$ an orthonormal set. If W_m denotes the closed subspace of $L^2(\mathbb{R};\mathbb{C})$ spanned by this set, then W_m is the orthogonal complement of V_m in V_{m-1}.*

Proof. If we define the sequence $\{\beta\}_{k\in\mathbb{Z}}$ by

$$\beta(\underline{k}) = (-1)^{\underline{k}}\overline{\alpha(1-\underline{k})}, \tag{4.20}$$

then we see that for each $\omega\in\mathbb{R}$ we have

$$\begin{aligned}\hat{\beta}(\omega) &= \sum_{k\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}2\pi\omega k}\mathrm{e}^{\mathrm{i}\pi k}\overline{\alpha(1-k)} \\ &= -\sum_{k\in\mathbb{Z}}\mathrm{e}^{2\pi\mathrm{i}\omega(1-k)}\mathrm{e}^{-\mathrm{i}2\pi\omega}\mathrm{e}^{2\pi\mathrm{i}(1-k)\frac{1}{2}}\overline{\alpha(1-k)} \\ &= -\mathrm{e}^{-\mathrm{i}2\pi\omega}\overline{\sum_{k\in\mathbb{Z}}\mathrm{e}^{-2\pi\mathrm{i}\omega(1-k)}\mathrm{e}^{-2\pi\mathrm{i}(1-k)\frac{1}{2}}\alpha(1-k)} \\ &= -\mathrm{e}^{-\mathrm{i}2\pi\bullet}\overline{\hat{\alpha}\left(\bullet+\tfrac{1}{2}\right)}.\end{aligned} \tag{4.21}$$

It is easy to check that (4.17) is a direct consequence of (4.11).

If we let $e_{\underline{k}} = \sqrt{2}\varphi(2\bullet+\underline{k})$, $u_{\underline{k}} = \varphi(\bullet+\underline{k})$, and $v_{\underline{k}} = \psi(\bullet+\underline{k})$, then it follows that (4.15) and (4.16) hold. Now we can apply Theorem 4.5 and do some easy calculations. □

Now we can finally give a basic result on wavelets.

Theorem 4.8. *Let $(\{V_m\}_{m\in\mathbb{Z}},\varphi)$ be a multiresolution of $L^2(\mathbb{R};\mathbb{C})$ and let ψ be the associated mother wavelet. Then the sets*

$$\{2^{m/2}\psi(2^m\bullet-k)\}_{m,k\in\mathbb{Z}}, \tag{4.22}$$

and

$$\{2^{m_0/2}\varphi(2^{m_0}\bullet-k), 2^{m/2}\psi(2^m\bullet-k)\}_{m\geq m_0,k\in\mathbb{Z}}, \tag{4.23}$$

where $m_0\in\mathbb{Z}$ is arbitrary, are orthonormal bases for $L^2(\mathbb{R};\mathbb{C})$.

Proof. It follows from theorem 4.7 that $V_m = V_j\oplus W_j\oplus W_{j+1}\oplus\ldots\oplus W_{m-1}$. Since $V_j\perp W_j$ we easily see that the sets in question are orthonormal and the fact that they span $L^2(\mathbb{R};\mathbb{C})$ is a consequence of (4.3) and proposition 4.3. □

2. Partitions of unity and $\lim_{m\to-\infty} V_m$

In this section we prove that the scaling function (or father wavelet) associated with a multiresolution gives a partition of unity when translated by integers. Here we have to introduce the additional restriction that φ belongs to $L^1(\mathbb{R})$, but since one usually wants the wavelets to have compact support or decay rapidly at infinity, this is not a serious restriction.

Theorem 4.9. *Let* (4.1), (4.3), *and* (4.4) *hold, and assume that* $\varphi \in L^1(\mathbb{R};\mathbb{C})$. *Then*

$$\left|\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x\right| = 1, \tag{4.24}$$

and

$$\sum_{k\in\mathbb{Z}} \varphi(\bullet - k) = \int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x, \tag{4.25}$$

where the series converges in $L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{C})$.

Usually one multiplies φ by a complex number with absolute value 1 so that $\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = 1$.

Proof. Let f_m be the orthogonal projection of the function $\chi_{[-4,4]}$ onto the space V_m. Then we have

$$f_m = \sum_{k\in\mathbb{Z}} 2^{-m} \int_{-4}^{4} \overline{\varphi(2^{-m}x - k)}\,\mathrm{d}x\ \varphi(2^{-m}\bullet - k). \tag{4.26}$$

Since $\lim_{m\to-\infty} V_m = L^2(\mathbb{R};\mathbb{C})$ it follows that $f_m \to \chi_{[-4,4]}$ in $L^2(\mathbb{R};\mathbb{C})$ and therefore also in $L^1_{\mathrm{loc}}(\mathbb{R})$.

Let $\gamma = \int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x$ and define the numbers $c_{k,m}$ by

$$c_{k,m} = \gamma - \int_{-2^{-m+2}-k}^{2^{-m+2}-k} \varphi(x)\,\mathrm{d}x, \quad k,m \in \mathbb{Z}. \tag{4.27}$$

Let b be the function $\sum_{k\in\mathbb{Z}} \varphi(\bullet - k)$. Since $\varphi \in L^1(\mathbb{R})$ it follows that b is locally integrable and periodic with period 1. Now we can rewrite the expression for f_m to be

$$f_m = \overline{\gamma} b(2^{-m}\bullet) - \sum_{k\in\mathbb{Z}} \overline{c_{k,m}} \varphi(2^{-m}\bullet - k). \tag{4.28}$$

If we can prove that

$$\lim_{m\to-\infty} \sum_{k\in\mathbb{Z}} |c_{k,m}| \int_{-1}^{1} |\varphi(2^{-m}x - k)|\,\mathrm{d}x = 0, \tag{4.29}$$

then it follows from (4.28) that

$$\lim_{m\to-\infty}\int_{-1}^{1}|\overline{\gamma}b(2^{-m}x)-1|\,\mathrm{d}x=\lim_{m\to-\infty}\int_{-1}^{1}|f_m(x)-1|\,\mathrm{d}x=0, \tag{4.30}$$

and it follows that $\overline{\gamma}b$ must be 1 almost everywhere. Hence we get $1 = \int_0^1 \overline{\gamma}b(x)\,\mathrm{d}x = \overline{\gamma}\int_{\mathbb{R}}\varphi(x)\,\mathrm{d}x = |\gamma|^2$. Thus $|\gamma| = 1$ and since $\overline{\gamma} = 1/\gamma$ we have all the desired conclusions.

Next, let us show that (4.29) holds. Clearly,

$$\begin{aligned}(4.31)\quad &\sum_{k\in\mathbb{Z}}|c_{k,m}|\int_{-1}^{1}|\varphi(2^{-m}x-k)|\,\mathrm{d}x\\ &=\sum_{p=0}^{2^{-m+1}-1}\sum_{q\in\mathbb{Z}}|c_{q2^{-m+1}+p,m}|2^m\int_{-2^{-m}-q2^{-m+1}+p}^{2^{-m}-q2^{-m+1}+p}|\varphi(x)|\,\mathrm{d}x\\ &=2^m\sum_{p=0}^{2^{-m+1}-1}\int_{\mathbb{R}}h_{p,m}(x)\,\mathrm{d}x\end{aligned}$$

where

$$h_{p,m}(x)=|c_{q2^{m+1}+p,m}\varphi(x)|\quad\text{when}\quad -2^m-q2^{m+1}-p\le x<2^m-q2^{m+1}-p.$$

Let $\epsilon > 0$ be arbitrary and choose $m < 0$ with $|m|$ to be so large that

$$\int_{|x|\ge 2^{-m}}|\varphi(x)|\,\mathrm{d}x<\frac{\epsilon}{4(1+\|\varphi\|_{L^1(\mathbb{R})})}.$$

Now

$$c_{k,m}=\int_{-\infty}^{-2^{-m}-k}\varphi(x)\,\mathrm{d}x+\int_{2^{-m}-k}^{\infty}\varphi(x)\,\mathrm{d}x, \tag{4.32}$$

so that is clear that $|c_{k,m}| \le \|\varphi\|_{L^1(\mathbb{R})}$. Thus we see that

$$\int_{|x|\ge 2^{-m}}h_{p,m}(x)\,\mathrm{d}x<\tfrac{\epsilon}{4}, \tag{4.33}$$

for all p. On the other hand we note that if $|x| < 2^{-m}$, then

$$h_{p,m}(x)\le|\varphi(x)|\max_{|k|\le 2^{-m+1}}|c_{k,m}|.$$

Since it follows from (4.32) that $|c_{k,m}| \le \int_{|x|\ge 2^{-m+1}}|\varphi(x)|\,\mathrm{d}x$ when $|k| \le 2^{-m+1}$ we easily see that

$$\int_{|x|<2^{-m}}h_{p,m}(x)\,\mathrm{d}x<\tfrac{\epsilon}{4}, \tag{4.34}$$

as well. Using (4.33) and (4.33) in (4.31) we see that $\sum_{k\in\mathbb{Z}}|c_{k,m}|\int_{-1}^{1}|\varphi(2^{-m}x-k)|\,\mathrm{d}x<\epsilon$, and the proof is completed. □

Next we consider the converse of this result.

Theorem 4.10. *Let* (4.1) *and* (4.4) *hold, and assume that*

$$\lim_{\substack{S\to-\infty\\ T\to+\infty}}\int_S^T \varphi(x)\,\mathrm{d}x=\gamma \quad \textit{where} \quad |\gamma|=1. \tag{4.35}$$

Then (4.3) *holds.*

Proof. We may multiply φ by $\overline{\gamma}$ and thus assume that $\gamma=1$. Let P_m denote the orthogonal projection on $L^2(\mathbb{R};\mathbb{C})$ onto V_m. The claim (4.3) is obviously equivalent to the fact that $P_mf\to f$ as $m\to-\infty$ for every $f\in L^2(\mathbb{R};\mathbb{C})$, and since $\|P_m\|=1$ and the space spanned by characteristic functions of intervals is dense in $L^2(\mathbb{R};\mathbb{C})$ it suffices to show that $P_m\chi_{[a,b]}\to\chi_{[a,b]}$ as $m\to\infty$ for arbitrary $a<b$. Since P_m is an orthogonal projection, we have

$$\begin{aligned}&\|\chi_{[a,b]}-P_m\chi_{[a,b]}\|^2=\|\chi_{[a,b]}\|^2-\|P_m\chi_{[a,b]}\|^2\\ &=b-a-\sum_{k\in\mathbb{Z}}\left|\int_a^b 2^{-\frac{m}{2}}\varphi(2^{-m}x-k)\,\mathrm{d}x\right|^2=b-a-\sum_{k\in\mathbb{Z}}2^m\left|\int_{2^{-m}a-k}^{2^{-m}b-k}\varphi(x)\,\mathrm{d}x\right|^2.\end{aligned}$$

Thus we see that if we can prove that

$$\liminf_{m\to\infty}\sum_{k\in\mathbb{Z}}2^m\left|\int_{2^{-m}a-k}^{2^{-m}b-k}\varphi(x)\,\mathrm{d}x\right|^2\ge b-a, \tag{4.36}$$

then we get the desired conclusion. Let $\epsilon>0$ be arbitrary and choose S_0 and T_0 to be such that if $S\le S_0$ and $T\ge T_0$, then

$$\left|\int_S^T\varphi(x)\,\mathrm{d}x\right|^2\ge 1-\epsilon. \tag{4.37}$$

Let I_m be the set

$$I_m=\{\,k\in\mathbb{Z}\mid 2^{-m}a-k\le S_0,\quad 2^{-m}b-k\ge T_0\,\}. \tag{4.38}$$

We clearly have

(4.39)
$$\sum_{k\in\mathbb{Z}}2^m\left|\int_{2^{-m}a-k}^{2^{-m}b-k}\varphi(x)\,\mathrm{d}x\right|^2\ge\sum_{k\in I_m}2^{-m}\left|\int_{2^{-m}a-k}^{2^{-m}b-k}\varphi(x)\,\mathrm{d}x\right|^2\ge 2^m\#I_m(1-\epsilon).$$

It is obvious that $\#I_m=\lfloor 2^{-m}b-T_0\rfloor-\lceil 2^{-m}a-S_0\rceil+1$ and therefore $2^{-m}\#I_m\to b-a$ as $m\to\infty$. But then we get (4.36) and the proof is completed. □

We need another version of this theorem as well:

Theorem 4.11. *Let* (4.1) *and* (4.4) *hold, and assume that* $\hat{\varphi}$ *is continuous at* 0 *and* $|\hat{\varphi}(0)| = 1$. *Then* (4.3) *holds.*

Proof. We may without loss of generality assume that $\hat{\varphi}(0) = 1$ since we may multiply φ by $\overline{\hat{\varphi}(0)}$.

Let P_m denote the orthogonal projection on $L^2(\mathbb{R};\mathbb{C})$ onto V_m. Since the claim (4.3) is equivalent to the fact that $P_m f \to f$ as $m \to -\infty$ for every $f \in L^2(\mathbb{R};\mathbb{C})$, and since $\|P_m\| = 1$ and the space $\mathcal{C}^\infty_\downarrow(\mathbb{R})$ is dense in $L^2(\mathbb{R};\mathbb{C})$ it suffices to show that $P_m f \to f$ as $m \to \infty$ for every $f \in \mathcal{C}^\infty_\downarrow(\mathbb{R})$. Since P_m is an orthogonal projection, we have

$$\|f - P_m f\|^2_{L^2(\mathbb{R})} = \|f\|^2_{L^2(\mathbb{R})} - \|P_m f\|^2_{L^2(\mathbb{R})}$$

and hence it suffices to show that

$$\liminf_{m\to\infty} \|P_m f\|^2_{L^2(\mathbb{R})} \geq \int_{\mathbb{R}} |f(t)|^2 \,\mathrm{d}t, \quad f \in C^\infty_\downarrow(\mathbb{R}). \tag{4.40}$$

Let $f \in \mathcal{C}^\infty_\downarrow(\mathbb{R}) \setminus \{0\}$. Since $(2^{-\frac{m}{2}}\varphi(2^{-m} \bullet - k))_{k\in\mathbb{Z}}$ is an orthonormal basis in V_m we have

$$\|P_m f\|^2_{L^2(\mathbb{R})} = \sum_{k\in\mathbb{Z}} 2^{-m} \left| \int_{\mathbb{R}} f(x)\overline{\varphi(2^{-m}x - k)}\,\mathrm{d}x \right|^2.$$

Next we use Plancherel's theorem, i.e., equation 2.5 and the fact that the Fourier transform of $\varphi(2^{-m} \bullet - k)$ is $2^m\hat{\varphi}(2^m\underline{\omega})\mathrm{e}^{-\mathrm{i}2\pi 2^m \underline{\omega} k}$ to get

$$\|P_m f\|^2_{L^2(\mathbb{R})} = \sum_{k\in\mathbb{Z}} 2^{m} \left| \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{\varphi}(2^m\omega)}\mathrm{e}^{\mathrm{i}2\pi 2^m k}\,\mathrm{d}\omega \right|^2 \tag{4.41}$$

The idea of the remaining part of the proof is that when $-m$ is sufficiently large, $\hat{\varphi}(2^m\omega) \approx 1$ and then we get by the Fourier inversion formula

$$\sum_{k\in\mathbb{Z}} 2^{m} \left| \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{\varphi}(2^m\omega)}\mathrm{e}^{\mathrm{i}2\pi 2^m k}\,\mathrm{d}\omega \right|^2 \approx \sum_{k\in\mathbb{Z}} 2^m |f(2^m k)|^2 \approx \int_{\mathbb{R}} |f(x)|^2 \,\mathrm{d}x.$$

It remains to get error estimates for these approximations.

Let $\epsilon > 0$. Since $f \in \mathcal{C}^\infty_\downarrow(\mathbb{R})$ there exists a positive integer n_0 such that if $m \leq 0$, then

$$\left| \int_{\mathbb{R}} |f(x)|^2 \,\mathrm{d}x - \sum_{k=-2^{-m+n_0}}^{2^{-m+n_0}} 2^m |f(2^m k)|^2 \right| < \frac{\epsilon}{2}. \tag{4.42}$$

We can choose a positive number ω_0 such that

$$\int_{|\omega|\geq\omega_0} |\hat{f}(\omega)| \,\mathrm{d}\omega < \frac{\epsilon}{\|\hat{f}\|_{L^1(\mathbb{R})} 2^{n_0+6}}. \tag{4.43}$$

Finally we choose $m_0 < 0$ such that then

$$\big|\hat{\varphi}(\eta) - 1\big| < \frac{\epsilon}{\|\hat{f}\|^2_{L^1(\mathbb{R})} 2^{n_0+5}}, \quad \text{if } |\eta| \le 2^{m_0}\omega_0. \tag{4.44}$$

Suppose now that $m \le m_0$. Then we get, for each $k \in \mathbb{Z}$, by the fact that $|\hat{\varphi}(2^m \underline{\omega})| \overset{\text{a.e.}}{\le} 1$ and by (4.43) and (4.44) that

$$\left| \int_{\mathbb{R}} f(\omega)\overline{\hat{\varphi}(2^m\omega)} e^{i2\pi 2^m k}\, d\omega - \int_{\mathbb{R}} f(\omega) e^{i2\pi 2^m k}\, d\omega \right|$$
$$\le 2 \int_{|\omega| \ge \omega} |\hat{f}(\omega)|\, d\omega + \sup_{|\eta| \le 2^m \omega_0} \big|\hat{\varphi}(\eta) - 1\big| \|\hat{f}\|_{L^1(\mathbb{R})} \le \epsilon \|\hat{f}\|_{L^1(\mathbb{R})} 2^{n_0+4}.$$

Taking into account the facts that $\left|\int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{\varphi}(2^m\omega)} e^{i2\pi 2^m k}\, d\omega\right| \le \|\hat{f}\|_{L^1(\mathbb{R})}$ and $\left|\int_{\mathbb{R}} f(\omega) e^{i2\pi 2^m k}\, d\omega\right| \le \|\hat{f}\|_{L^1(\mathbb{R})}$ we conclude that

$$\left| \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{\varphi}(2^m\omega)} e^{i2\pi 2^m k}\, d\omega \right|^2 \ge \left| \int_{\mathbb{R}} \hat{f}(\omega) e^{i2\pi 2^m k}\, d\omega \right|^2 - \frac{\epsilon}{2^{n_0+3}}.$$

and then by the Fourier inversion theorem 2.3 we have

$$\sum_{k\in\mathbb{Z}} 2^m \left| \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{\varphi}(2^m\omega)} e^{i2\pi 2^m k}\, d\omega \right|^2$$
$$\ge \sum_{k=-2^{-m+n_0}}^{2^{-m+n_0}} 2^m \left|f(2^m k)\right|^2 - 2^m(2^{-m+n_0+1} + 1)\frac{\epsilon}{2^{n_0+3}}.$$

Combining this inequality with (4.41) and (4.42) we finally get

$$\|P_m f\|^2_{L^2(\mathbb{R})} \ge \int_{\mathbb{R}} |f(x)|^2\, dx - \epsilon$$

and since ϵ was arbitrary we get the desired conclusion and the proof is completed.

□

3. Filters that determine multiresolutions

In this section we take as a starting point equation (4.9). Assume that a sequence $\alpha \in l^1(\mathbb{Z};\mathbb{C}) \subset l^2(\mathbb{Z};\mathbb{C})$ is given such that $\sum_{k\in\mathbb{Z}} \alpha(k) = 1$. Now the question to be studied is when this sequence is the filter for a multiresolution. A number of conditions are immediately obvious, and some other are less obvious.

Theorem 4.12. *Let the sequence $\{\alpha(k)\}_{k\in\mathbb{Z}}$ satisfy the following conditions.*

$$\sum_{k\in\mathbb{Z}} \log(|k|+1)|\alpha(k)| < \infty. \tag{4.45}$$

$$\sum_{k\in\mathbb{Z}} \alpha(k) = 1. \tag{4.46}$$

$$|\hat{\alpha}(\bullet)|^2 + \left|\hat{\alpha}\left(\bullet + \tfrac{1}{2}\right)\right|^2 = 1 \quad a.e.\ on \quad \mathbb{R}. \tag{4.47}$$

(4.48) *There exists a bounded Borel set $\mathcal{G} \subset \mathbb{R}$ such that $\sum_{k\in\mathbb{Z}} \chi_{\mathcal{G}}(\omega + k) = 1$ for all $\omega \in \mathbb{R}$ and $\hat{\alpha}$ does not vanish on the set $\overline{\bigcup_{k=1}^{\infty}(2^{-k}\mathcal{G})}$.*

Then there exists a multiresolution $(\{V_m\}_{m\in\mathbb{Z}}, \varphi)$ of $L^2(\mathbb{R};\mathbb{C})$ such that α is the associated filter.

Here $2^{-k}\mathcal{G}$ is, of course, the set $\{\, 2^{-k}\omega \mid \omega \in \mathcal{G}\,\}$.

Proof. Define the function h_m for each $m \in \mathbb{N}$ to be

$$h_m = \prod_{k=1}^{m} \hat{\alpha}(2^{-k}\bullet)\chi_{2^m\mathcal{G}}. \tag{4.49}$$

Since $\hat{\alpha}(0) = 1$ we may clearly assume that $0 \in \mathrm{i}nt(\mathcal{G})$ and therefore it follows from (4.45) and Lemma 5.1 (consider α as a measure supported on the integers) that h_m converges uniformly on compact sets to the function

$$h = \prod_{k=1}^{\infty} \hat{\alpha}(2^{-k}\bullet). \tag{4.50}$$

Since $\hat{\alpha}$ is continuous it follows from the uniform convergence that h is continuous as well.

An immediate consequence of assumption (4.48) is that if $f \in L^1(\mathbb{T};\mathbb{C})$ (that is, $f \in L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{C})$ is periodic with period 1), then

$$\int_{\mathcal{G}} f(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x. \tag{4.51}$$

Let $k \in \mathbb{Z}$ be arbitrary. Next we evaluate the integral $\int_{\mathbb{R}} |h_m(\omega)|^2 \mathrm{e}^{\mathrm{i}2\pi k\omega}\,\mathrm{d}\omega$ for arbitrary $m \in \mathbb{N}$. Using (4.11), (4.51) and the fact that $\hat{\alpha}$ and $\mathrm{e}^{-\mathrm{i}2\pi\bullet}$ are periodic with period 1 we get

$$\begin{aligned}
\int_{\mathbb{R}} |h_m(\omega)|^2 \mathrm{e}^{\mathrm{i}2\pi k\omega}\,\mathrm{d}\omega &= \int_{2^m\mathcal{G}} \prod_{j=1}^{m} \left|\hat{\alpha}\left(2^{-j}\omega\right)\right|^2 \mathrm{e}^{\mathrm{i}2\pi k\omega}\,\mathrm{d}\omega \\
&= 2^m \int_{\mathcal{G}} \prod_{j=0}^{m-1} \left|\hat{\alpha}(2^j\omega)\right|^2 \mathrm{e}^{\mathrm{i}2\pi 2^m k\omega}\,\mathrm{d}\omega \\
&= 2^m \int_0^1 \prod_{j=0}^{m-1} \left|\hat{\alpha}(2^j\omega)\right|^2 \mathrm{e}^{\mathrm{i}2\pi 2^m k\omega}\,\mathrm{d}\omega \\
&= 2^m \int_0^{1/2} \prod_{j=1}^{m-1} \left|\hat{\alpha}(2^j\omega)\right|^2 \left(|\hat{\alpha}(\omega)|^2 + \left|\hat{\alpha}\left(\omega+\tfrac{1}{2}\right)\right|^2\right) \mathrm{e}^{\mathrm{i}2\pi 2^m k\omega}\,\mathrm{d}\omega \\
&= 2^{m-1} \int_0^1 \prod_{j=0}^{m-2} \left|\hat{\alpha}(2^j\omega)\right|^2 \mathrm{e}^{\mathrm{i}2\pi 2^{m-1} k\omega}\,\mathrm{d}\omega \\
&= \int_{\mathbb{R}} |h_{m-1}(\omega)|^2 \mathrm{e}^{\mathrm{i}2\pi k\omega}\,\mathrm{d}\omega.
\end{aligned}$$

Since $\int_{\mathbb{R}} |h_0(\omega)|^2 \mathrm{e}^{\mathrm{i}2\pi k\omega}\,\mathrm{d}\omega = \int_{\mathcal{G}} \mathrm{e}^{\mathrm{i}2\pi k\omega}\,\mathrm{d}\omega = \delta_{0,k}$ it follows by induction that

$$\int_{\mathbb{R}} |h_m(\omega)|^2 \mathrm{e}^{\mathrm{i}2\pi k\omega}\,\mathrm{d}\omega = \delta_{0,k}, \tag{4.52}$$

and, in particular that

$$\|h_m\|_{L^2(\mathbb{R})} = 1. \tag{4.53}$$

Thus we also get with the aid of Fatou's lemma that $h \in L^2(\mathbb{R};\mathbb{C})$ and $\|h\|_{L^2(\mathbb{R})} \le 1$.

The function h is continuous and does not vanish on $\overline{\mathcal{G}}$, therefore there is a constant C such that $|h(x)| \ge C > 0$ for all $x \in \mathcal{G}$. Because h_m vanishes outside $2^m\mathcal{G}$ and satisfies $h_m = h/h(2^{-m}\bullet)$ on $2^m\mathcal{G}$ it follows that

$$|h_m(x)|^2 \le C^{-2}|h(x)|^2, \quad x \in \mathbb{R}. \tag{4.54}$$

This inequality allows us to apply the dominated convergence theorem and we conclude that

$$h_m \to h \quad \text{in} \quad L^2(\mathbb{R};\mathbb{C}). \tag{4.55}$$

If we define $\varphi \in L^2(\mathbb{R};\mathbb{C})$ by $\hat{\varphi} = h$, then it follows from (4.52) that

$$\int_{\mathbb{R}} |\hat{\varphi}(\omega)|^2 \mathrm{e}^{\mathrm{i}2\pi k\omega}\,\mathrm{d}\omega = \delta_{0,k}, \tag{4.56}$$

and by lemma 4.2 this is equivalent to the fact that

$$\{\varphi(\bullet - k)\}_{k\in\mathbb{Z}} \quad \text{is an orthonormal set in} \quad L^2(\mathbb{R};\mathbb{C}). \tag{4.57}$$

It follows immediately from (4.50) that (4.10) holds, and therefore we also have (4.9). But then we get (4.2). Finally, (4.3) follows from Theorem 4.11 and the fact that $\hat{\varphi}$ is continuous with $\hat{\varphi}(0) = 1$ by definition. □

If we require somewhat more of the scaling function, then we can get necessary and sufficient conditions.

Theorem 4.13. *Let $M \geq 1$. Then*

(4.58) *there is a multiresolution $(\{V_m\}_{m\in\mathbb{Z}}, \varphi)$ of $L^2(\mathbb{R};\mathbb{C})$ with filter α and $|\bullet|^M\varphi(\bullet) \in L^2(\mathbb{R};\mathbb{C})$*

if and only if

(4.59) *the sequence α satisfies $\bullet^M\alpha(\bullet) \in \ell^2(\mathbb{Z};\mathbb{C})$, and (4.11), (4.46), and (4.48) hold.*

If these conditions hold, then it follows that $\bullet^M\psi(\bullet) \in L^2(\mathbb{R};\mathbb{C})$ as well.

Proof. Assume that $(\{V_m\}_{m\in\mathbb{Z}}, \varphi)$ is a multiresolution of $L^2(\mathbb{R};\mathbb{C})$ with filter α and $|\bullet|^M\varphi(\bullet) \in L^2(\mathbb{R};\mathbb{C})$. It follows immediately from theorem 2.8 that $\hat{\varphi} \in H^M(\mathbb{R};\mathbb{C})$.

First we prove the series $\sum_{k\in\mathbb{Z}}|\hat{\varphi}(\omega + k)|^2$ converges uniformly for all $\omega \in [0, 1]$. Since

$$\hat{\varphi}(\omega) = \hat{\varphi}(\eta) + \int_{\omega}^{\eta} \hat{\varphi}'(\xi)\,\mathrm{d}\xi, \quad \omega, \eta \in [0, 1], \tag{4.60}$$

we have

$$|\hat{\varphi}(\omega + k)|^2 \leq 2\int_0^1 |\hat{\varphi}(\eta + k)|^2\,\mathrm{d}\eta + 2\int_0^1 |\hat{\varphi}'(\xi + k)|^2\,\mathrm{d}\xi. \tag{4.61}$$

Since $\hat{\varphi}$ and $\hat{\varphi}' \in L^2(\mathbb{R};\mathbb{C})$ we get the desired result, in particular, by (4.4) and lemma 4.2 that

$$\sum_{k\in\mathbb{Z}}|\hat{\varphi}(\omega + k)|^2 = 1, \quad \omega \in \mathbb{R}. \tag{4.62}$$

It follows that for each $\omega \in [0, 1]$ there exist numbers $k_\omega \in Z$ and $\epsilon_\omega > 0$ such that $|\hat{\varphi}(\xi + k_\omega)| > 0$ when $|\xi - \omega| < \epsilon_\omega$. Since $[0, 1]$ is compact we can choose finitely many of these points ω_j, $j = 1, 2, \ldots, n$ such that $[0, 1] \subset \bigcup_{j=1}^n(\omega_j - \epsilon_{\omega_j}, \omega_j + \epsilon_{\omega_j})$. But then we can construct the set $\mathcal{G}$ as the finite union of halfopen intervals. Since it follows from (4.10) that

$$\hat{\varphi}(\omega) = \hat{\varphi}(2^{-k}\bullet)\prod_{j=1}^{k}\hat{\alpha}(2^{-j}\bullet) \tag{4.63}$$

it follows from that fact that $\hat{\varphi}$ does not vanish on $\overline{\mathcal{G}}$ that $\hat{\alpha}$ cannot vanish on $\overline{\bigcup_{k=1}^{\infty}(2^{-k}\mathcal{G})}$. Thus we have established (4.48).

But it is also a consequence of the argument above that there exists a constant $C > 0$ such that $|\hat{\varphi}(\xi + k_{\omega_j})| \geq C$ when $|\xi - \omega_j| < \epsilon_{\omega_j}$ for $j = 1, \dots, n$. But since $\hat{\alpha}(\xi) = \hat{\varphi}(2(\xi + k_{\omega_j}))/\hat{\varphi}(\xi + k_{\omega_j})$ when $|\xi - \omega_j| < \epsilon_{\omega_j}$, we conculude that we $\hat{\alpha}$ is $M-1$ times continuously differentiable, and $\hat{\alpha}^{(M-1)}$ is absolutely continuous with a square integrable derivative on the interval $(\omega_j - \epsilon_{\omega_j}, \omega_j + \epsilon_{\omega_j})$. But since $[0,1] \subset \bigcup_{j=1}^{n}(\omega_j - \epsilon_{\omega_j}, \omega_j + \epsilon_{\omega_j})$ we conclude that $\hat{\alpha} \in H^M(\mathbb{T};\mathbb{C})$. But then it follows from theorem 2.8 that $\bullet^M \alpha \in \ell^2(\mathbb{Z};\mathbb{C})$.

It is a consequence of theorem 4.9 that $|\hat{\varphi}(0)| = 1$. It follows that we have $\hat{\alpha}(0) = \hat{\varphi}(2 \cdot 0)/\hat{\varphi}(0) = 1$, and since $\alpha \in \ell^1(\mathbb{Z};\mathbb{C})$ we get (4.46). This complete the first part of the proof since it follows from Theorem 4.4 that (4.11) holds true.

Next we assume that $\bullet^M \alpha(\bullet) \in \ell^2$, and (4.11), (4.46), and (4.48) hold. We know from theorem 4.12 that α is the filter associated with a multiresolution having scaling function φ. Since $\hat{\alpha}(0) = |\hat{\varphi}(0)| = 1$ there exists a number $\delta > 0$ such that $|\hat{\alpha}(\omega)| \geq 1/2$ and $|\hat{\varphi}(\omega)| \geq 1/2$ when $|\omega| \leq \delta$. But then it follows that $b \stackrel{\text{def}}{=} \log(\hat{\alpha})$ belongs to $H^M([-\delta,\delta];\mathbb{C})$ and an easy calculation shows that

$$\sup_{m \geq 1} \|\sum_{k=1}^{m} b(2^{-k}\bullet)\|_{H^M([-\delta,\delta])} < \infty. \tag{4.64}$$

Since $\sum_{k=1}^{m} b(2^{-k}\omega) \to \log(\hat{\varphi}(\omega))$ as $m \to \infty$ when $|\omega| \leq \delta$, we conclude that $\log(\hat{\varphi})$ and therefore also $\hat{\varphi}$ belong to $H^M([-\delta,\delta];\mathbb{C})$. Using an induction argument and the formula $\hat{\varphi}(2\bullet) = \hat{\alpha}\hat{\varphi}$ we can show that $\hat{\varphi} \in H^M([-2^j\delta, 2^j\delta];\mathbb{C})$ for each $j \geq 0$.

Now we have to prove that $\hat{\varphi} \in H^M(\mathbb{R};\mathbb{C})$ and by using an induction argument, we may assume that $\hat{\varphi} \in H^{M-1}(\mathbb{R};\mathbb{C})$. Recall also that $\sup_{\omega \in \mathbb{R}} |\hat{\alpha}^{(j)}(\omega)| < \infty$ for $j = 0, 1, \dots, M-1$ and that in particular $|\hat{\alpha}(\omega)| \leq 1$. If we differentiate both sides of the equation (4.10) M times we conclude that there exists a constant C such that for every positive integer k we have

$$\begin{aligned} &2^M \|\hat{\varphi}^{(M)}(2\bullet)\|_{L^2([-k,k])} \\ &\quad \leq C + \sqrt{\int_{-k}^{k} |\hat{\alpha}^{(M)}(\omega)|^2 |\hat{\varphi}(\omega)|^2 \,\mathrm{d}\omega} + \sqrt{\int_{-k}^{k} |\hat{\alpha}(\omega)|^2 |\hat{\varphi}^{(M)}(\omega)|^2 \,\mathrm{d}\omega} \end{aligned}$$

and therefore

$$\begin{aligned}
2^{M-1/2}\sqrt{\int_{-k}^{k}|\hat\varphi^{(M)}(\omega)|^2\,\mathrm{d}\omega} &\le 2^{M-1/2}\sqrt{\int_{-2k}^{2k}|\hat\varphi^{(M)}(\omega)|^2\,\mathrm{d}\omega}\\
&\le C+\sqrt{\int_0^1|\hat\alpha^{(M)}(\omega)|^2\sum_{j=-k}^{k-1}|\hat\varphi(\omega+j)|^2\,\mathrm{d}\omega}+\sqrt{\int_{-k}^{k}|\hat\varphi^{(M)}(\omega)|^2\,\mathrm{d}\omega}\\
&\le C+\sqrt{\int_0^1|\hat\alpha^{(M)}(\omega)|^2\,\mathrm{d}\omega}+\sqrt{\int_{-k}^{k}|\hat\varphi^{(M)}(\omega)|^2\,\mathrm{d}\omega}.
\end{aligned}$$

Since $M \ge 1$ we conclude that

$$\sup_{k\ge 1}\int_{-k}^{k}|\hat\varphi^{(M)}(\omega)|^2\,\mathrm{d}\omega<\infty, \tag{4.65}$$

which is what we want to prove. This completes the second part of the proof.

Finally we have to prove that if $\hat\varphi \in H^M(\mathbb{R};\mathbb{C})$, then we also have $\hat\psi \in H^M(\mathbb{R};\mathbb{C})$ and this can be done with the aid of the equation

$$\hat\psi(2\bullet)=\hat\beta(\bullet)\varphi(\bullet). \tag{4.66}$$

and an argument related to but easier than the one used above. □

It is quite clear that if $\hat\alpha$ does not vanish on $[-\frac14,\frac14]$, then (4.48) holds because we can take $\mathcal{G}=[-\frac12,\frac12)$. We can however improve this result slightly, provided that we assume (4.11).

Proposition 4.14. *Assume that $\alpha \in \ell^1(\mathbb{Z};\mathbb{C})$ satisfies* (4.11) *and $\hat\alpha$ does not vanish on the interval $[-\frac16,\frac16]$. Then* (4.48) *holds true.*

Proof. Since $\alpha \in \ell^1(\mathbb{Z};\mathbb{C})$ we know that $\hat\alpha \in C(\mathbb{T};\mathbb{C})$. By (4.11) we know that $\max\{|\hat\alpha(\bullet)|,|\hat\alpha(\bullet+\frac12)|\}\ge\delta>0$ (where, in fact $\delta=1/\sqrt{2}$).

We construct $\mathcal{G}$ e.g. as follows. Let $k_1=0$ if $|\hat\alpha(\frac16)|\ge\delta$ and $k_1=1$ otherwise (in which case $|\hat\alpha(-\frac13)|\ge\delta$) and let $k_{j+1}=1-k_j$ for $j\ge1$. Choose $a_1=\frac13$ and

$$a_{j+1}=\min\{\inf\{\,\omega\mid\omega>a_j,\quad \big|\hat\alpha\big(\tfrac{\omega-k_j}{2}\big)\big|\le\tfrac{\delta}{2}\,\},\tfrac23\}. \tag{4.67}$$

It follows from this construction that $|\hat\alpha(\frac{a_j-k_j}{2})|\ge\delta$ for all j and from the continuity of $\hat\alpha$ we therefore get that there exists a number n such that $a_{n+1}=\frac23$. Now we define

$$\mathcal{G}=\bigcup_{j=1}^{n}[a_j-k_j,a_{j+1}-k_j)\bigcup\Big[-\tfrac13,\tfrac13\Big). \tag{4.68}$$

This set has all the desired properties because $\bigcup_{k=2}^{\infty} 2^{-k}\mathcal{G} \subset [-\frac{1}{6}, \frac{1}{6}]$. □

As we see from the proof above, it is not essential that we have the full force of (4.11) it suffices to assume that $\max\{|\hat{\alpha}(\bullet)|, |\hat{\alpha}(\bullet + \frac{1}{2})|\} \geq \delta > 0$.

On the other hand, we can also give conditions that guarantee that (4.48) is not satisfied. Moreover, it follows from this result that there is no hope that one could replace the interval $[-\frac{1}{6}, \frac{1}{6}]$ by a smaller one. We shall also use this result later when proving that the only sacling function φ that has compact support and is symmetric is the Haar function.

Proposition 4.15. *Assume that $\alpha \in \ell^1(\mathbb{Z}; \mathbb{C})$ is such that for some integer $m \geq 1$*

$$\hat{\alpha}\Big(\frac{2k+1}{2(2m+1)}\Big) = 0, \quad k \in \mathbb{Z}. \tag{4.69}$$

Then (4.48) *does not hold.*

It is not really essential that $\alpha \in \ell^1(\mathbb{Z}; \mathbb{C})$, what we need is that $\hat{\alpha}$ is continuous.

Proof. If (4.48) holds, then there exists an integer j such that $\frac{1}{2m+1} + j \in \mathcal{G}$. Because $m \neq 0$ we get

$$\tfrac{1}{2m+1} + j = \tfrac{2^p(2k+1)}{2m+1}, \tag{4.70}$$

where $p \geq 1$. But then

$$\hat{\alpha}\Big(2^{-p}\Big(\tfrac{1}{2m+1} + j\Big)\Big) = 0, \tag{4.71}$$

and this contradicts (4.48). □

An example where the result above is applicable is the filter $\alpha(-1) = \alpha(2) = \frac{1}{2}$ and $\alpha(k) = 0$ otherwise. Then one easily sees that $\hat{\alpha} = \cos(3\pi\bullet)e^{-\pi i \bullet}$ and (4.11) is satisfied but (4.48) fails by the result above.