Fourier transforms

1. Basic definitions

In this section we introduce some basic definitions of Fourier transforms. Instead of formulating three different results for the cases we consider here, one could obtain much greater generality by considering functions defined on some Abelian group. There would be some differences depending on whether the group is compact or not but otherwise one could have a unified treatment. We will, however, not take this approach here.

Note also that there are several alternatives for where the factor 2π will appear, the choice of having it in the exponent is only one. We denote $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, that is a function f defined on $\mathbb{T}$ can be thought of as a function defined on $\mathbb{R}$ with the property that $f(t+1) = f(t)$ for all $t \in \mathbb{R}$. The spaces $L^p(\mathbb{T};\mathbb{C})$, $1 \le p \le \infty$, can therefore be indentified with the spaces $L^p([0,1];\mathbb{C})$, but $\mathcal{C}(\mathbb{T};C)$ is not the same space as $\mathcal{C}([0,1];\mathbb{C})$.

Definition 2.1. *If $f \in L^1(\mathbb{T};\mathbb{C})$, (or $f \in L^1([0,1];\mathbb{C})$) then its Fourier transform is the sequence $\hat{f}$ defined by*

$$\hat{f}(k) = \int_0^1 \mathrm{e}^{-\mathrm{i}2\pi kt} f(t)\,\mathrm{d}t, \quad k \in \mathbb{Z}. \tag{2.1}$$

If $F \in L^1(\mathbb{R};\mathbb{C})$, then its Fourier transform is the function $\hat{F}$ defined by

$$\hat{F}(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi\omega t} F(t)\,\mathrm{d}t, \quad \omega \in \mathbb{R}. \tag{2.2}$$

If $\phi \in \ell^1(\mathbb{Z};\mathbb{C})$, then its Fourier transform is the function $\hat{\phi}$ defined by

$$\hat{\phi}(\omega) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{-2\pi\mathrm{i}\omega k} \phi_k, \quad \omega \in \mathbb{R}. \tag{2.3}$$

The most basic properties of the Fourier transform are given by the so-called Riemann-Lebesgue lemma.

Theorem 2.2.

(a) *If* $f \in L^1(\mathbb{T};\mathbb{C})$, *then* $\hat{f} \in c_0(\mathbb{Z};\mathbb{C})$.

(b) *If* $F \in L^1(\mathbb{R};\mathbb{C})$, *then* $\hat{F} \in C_0(\mathbb{R};\mathbb{C})$.

(c) *If* $\phi \in \ell^1(\mathbb{Z};\mathbb{C})$, *then* $\hat{\phi} \in C(\mathbb{T};\mathbb{C})$.

On a general level, it is quite easy to describe how to invert the Fourier transform, the details of exactly what assumptions are needed can be much more complicated.

Theorem 2.3.

(a) *If* $f \in L^1(\mathbb{T};\mathbb{C})$ *and* $\hat{f} \in \ell^1(\mathbb{Z};\mathbb{C})$ *then* $f(x) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi kx}\hat{f}(k)$, *for a.e.* $x \in [0,1]$.

(b) *If* $F \in L^1(\mathbb{R};\mathbb{C})$ *and* $\hat{F} \in L^1(\mathbb{R};\mathbb{C})$ *then* $F(x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi x\omega}\hat{F}(\omega)\,\mathrm{d}\omega$, *for a.e.* $x \in \mathbb{R}$.

(c) *If* $\phi \in \ell^1(\mathbb{Z};\mathbb{C})$ *and* $\hat{\phi} \in L^1(\mathbb{T};\mathbb{C})$ *then* $\phi_k = \int_0^1 \mathrm{e}^{\mathrm{i}2\pi k\omega}\hat{\phi}(\omega)\,\mathrm{d}\omega$, *for all* $k \in \mathbb{Z}$.

These inversion formulas hold in much greater generality, but then more care has to be spent on formulating in exactly what sense the Fourier transform and the inversion formula hold.

An important part of Fourier analysis is the extension to square integrable functions. Here we should of course observe that $L^2(\mathbb{T};\mathbb{C}) \subset L^1(\mathbb{T};\mathbb{C})$ (that is, $L^2([0,1];\mathbb{C}) \subset L^1([0,1];\mathbb{C})$), $\ell^1(\mathbb{Z};\mathbb{C}) \subset \ell^2(\mathbb{Z};\mathbb{C})$, $L^1(\mathbb{R};\mathbb{C}) \not\subset L^2(\mathbb{R};\mathbb{C})$ and $L^2(\mathbb{R};\mathbb{C}) \not\subset L^1(\mathbb{R};\mathbb{C})$, but $L^1(\mathbb{R};\mathbb{C}) \bigcap L^2(\mathbb{R};\mathbb{C})$ is dense in $L^2(\mathbb{R};\mathbb{C})$.

Theorem 2.4.

(a) *The Fourier transform is an isometric isomorphism from* $L^2(\mathbb{T};\mathbb{C})$ *to* $\ell^2(\mathbb{Z};\mathbb{C})$, *that is* $\|f\|_{L^2(\mathbb{T})} = \|\hat{f}\|_{\ell^2(\mathbb{Z})}$ *for every* $f \in L^2(\mathbb{T};\mathbb{C})$. *In particular, if* f *and* $g \in L^2(\mathbb{T};\mathbb{C})$, *then*

$$\int_0^1 f(x)\overline{g(x)}\,\mathrm{d}x = \sum_{k\in z} \hat{f}(k)\overline{\hat{g}(k)}. \tag{2.4}$$

(b) *The Fourier transform can be extended to an isometric isomorphism on* $L^2(\mathbb{R};\mathbb{C})$, *that is* $\|F\|_{L^2(\mathbb{R})} = \|\hat{F}\|_{L^2(\mathbb{R})}$ *for every* $F \in L^2(\mathbb{R};\mathbb{C})$. *In* *particular, if F and $G \in L^2(\mathbb{R};\mathbb{C})$, then*

$$\int_{\mathbb{R}} F(x)\overline{G(x)}\,\mathrm{d}x = \int_{\mathbb{R}} \hat{F}(\omega)\overline{\hat{G}(\omega)}\,\mathrm{d}\omega. \tag{2.5}$$

(c) *The Fourier transform can be extended to an isometric isomorphism from $\ell^2(\mathbb{Z};\mathbb{C})$ to $L^2(\mathbb{T};\mathbb{C})$, that is $\|\phi\|_{L^2(\mathbb{R})} = \|\hat{\phi}\|_{L^2([0,1])}$ for every $\phi \in \ell^2(\mathbb{Z};\mathbb{C})$. In particular, if ϕ and $\gamma \in \ell^2(\mathbb{Z};\mathbb{C})$, then*

$$\sum_{k\in\mathbb{Z}} \phi_k\overline{\gamma_k} = \int_0^1 \hat{\phi}(\omega)\overline{\hat{\gamma}(\omega)}\,\mathrm{d}\omega. \tag{2.6}$$

Once one has defined the Fourier transform for both integrable and square integrable functions it is obvious that the Fourier transform is also well-defined on $L^1(\mathbb{R};\mathbb{C}) + L^2(\mathbb{R};\mathbb{C})$.

Next we give another result on the inversion of Fourier transforms. There are many different versions of the inversion formula that one could consider here, but we concentrate on the results that we will need and use.

Theorem 2.5.

(a) *Assume that $x_0 \in [0,1]$ and $f \in L^1(\mathbb{T};\mathbb{C})$ are such that*

$$\frac{f(\bullet) - f(x_0)}{\bullet - x_0} \in L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{C}). \tag{2.7}$$

Then

$$f(x_0) = \lim_{\substack{k\to-\infty\\ m\to\infty}} \sum_{j=k}^{m} \mathrm{e}^{\mathrm{i}2\pi x_0 j}\hat{f}(j). \tag{2.8}$$

(b) *Assume that $x_0 \in \mathbb{R}$ and $F \in L^1(\mathbb{R};\mathbb{C}) + L^2(\mathbb{R};\mathbb{C})$ are such that*

$$\frac{F(\bullet) - F(x_0)}{\bullet - x_0} \in L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{C}). \tag{2.9}$$

Then

$$F(x_0) = \lim_{\substack{S\to-\infty\\ T\to+\infty}} \int_S^T \mathrm{e}^{\mathrm{i}2\pi x_0\omega}\hat{F}(\omega)\,\mathrm{d}\omega. \tag{2.10}$$

Proof. (a) Let g be a function defined to be

$$g = \frac{f(\bullet) - f(x_0)}{\mathrm{e}^{-\mathrm{i}2\pi\bullet} - \mathrm{e}^{-\mathrm{i}2\pi x_0}}. \tag{2.11}$$

It follows from our assumption that $g \in L^1(\mathbb{T};\mathbb{C})$. A straightforward calculation shows that

$$\hat{f}(j) = f(x_0)\delta_{0,j} + \hat{g}(j+1) - \mathrm{e}^{-\mathrm{i}2\pi x_0}\hat{g}(j), \tag{2.12}$$

and this means that if $k < 0$ and $m > 0$, then

$$\sum_{j=k}^{m} e^{i2\pi x_0 j}\hat{f}(j) = f(x_0) + e^{-i2\pi x_0}\sum_{j=k}^{m}\Big(e^{i2\pi x_0(j+1)}\hat{g}(j+1) - e^{i2\pi x_0 j}\hat{g}(j)\Big)$$
$$= f(x_0) + e^{i2\pi x_0 m}\hat{g}(m+1) - e^{i2\pi x_0(k-1)}\hat{g}(k).$$

Now we get the desired conclusion because it follows from theorem 2.2 and from the fact that $g \in L^1(\mathbb{T};\mathbb{C})$ that $\hat{g}(j) \to 0$ as $|j| \to \infty$.

(b) Let $G = F(x_0)e^{-\pi(\bullet^2 - x_0^2)}$ so that $\hat{G} = F(x_0)e^{-\pi(\bullet^2 - x_0^2)}$ as well (any sufficiently nice function would do here). Since $\hat{G}$ is integrable, the lemma holds with F replaced by G by theorem 2.3 and since G is differentiable we conclude that by taking $F - G$ instead of F, we may, without loss of generality, assume that $F(x_0) = 0$.

Applying (2.5) with $\hat{G} = e^{-i2\pi x_0 \bullet}\chi_{[S,T]}$ we get

(2.13)
$$\int_S^T e^{i2\pi x_0\omega}\hat{F}(\omega)\,d\omega = \int_{\mathbb{R}} F(x)\frac{1}{i2\pi(x-x_0)}\left(e^{i2\pi(x-x_0)S} - e^{i2\pi(x-x_0)T}\right)dx.$$

Now we observe that it follows from our assumptions that

$$\int_{\mathbb{R}}\left|F(x)\frac{1}{i2\pi(x-x_0)}\right|dx < \infty \tag{2.14}$$

and therefore the desired conclusion follows from theorem 2.2.(b). □

We need a result on the Fourier transform of rapidly decaying infinitely differentiable functions as well.

Definition 2.6.

$$\mathcal{C}_{\downarrow}^{\infty}(\mathbb{R}) = \mathcal{S}(\mathbb{R}) = \left\{ f : \mathbb{R} \mapsto \mathbb{C} \;\middle|\; f \in C^{\infty}(\mathbb{R}),\quad t^k f^{(n)}(t) \in L^{\infty}(\mathbb{R}) \text{ for all } k, n \geq 0 \right\}.$$

Thus $\mathcal{C}_{\downarrow}^{\infty}(\mathbb{R})$ consists of all infinitely differentiable functions whose derivatives converge to zero faster than every function of the form $|t|^{-n}$ as $|t| \to \infty$. Since $\mathcal{C}_{\downarrow}^{\infty}(\mathbb{R})$ is a subspace of $L^1(\mathbb{R})$, the Fourier transform is weel defined. Furthermore, one has:

Theorem 2.7. *The Fourier transform "$\hat{\ }$" is a bijection:* $\mathcal{C}_{\downarrow}^{\infty}(\mathbb{R}) \mapsto \mathcal{C}_{\downarrow}^{\infty}(\mathbb{R})$.

In the square integrable (summable) case there is a direct connection with differentiability of the transform:

Theorem 2.8. *Let* $M \in \mathbb{N}$.

(a) *Let* $F \in L^2(\mathbb{R};\mathbb{C})$. *Then* $|\bullet|^M F(\bullet) \in L^2(\mathbb{R};\mathbb{C})$ *if and only if* $\hat{F}^{(j)}$ *is locally absolutely continuous for* $j = 0,\dots,M-1$ *and* $\hat{F}^{(j)} \in L^2(\mathbb{R};\mathbb{C})$ *for* $j = 0,\dots,M$.

(b) *Let* $\phi \in \ell^2(\mathbb{Z};\mathbb{C})$. *Then* $|\bullet|^M \phi(\bullet) \in \ell^2(\mathbb{R};\mathbb{C})$ *if and only if* $\hat{\phi}^{(j)}$ *is absolutely continuous for* $j = 0,\dots,M-1$ *and* $\hat{\phi}^{(j)} \in L^2(\mathbb{T};\mathbb{C})$ *for* $j = 0,\dots,M$ *(or equivalently just* $\hat{\phi}^{(M)} \in L^2(\mathbb{T};\mathbb{C})$*).*