Bases and frames

1. Bases in Banach spaces

Definition 3.1. *Let V be a Banach space with norm $\|\cdot\|$. A sequence $(g_j)_{j=1}^{\infty} \subset V$ is a Schauder basis for V provided that for each element $x \in V$ there exist unique coefficients γ_j such that $\sum_{j=1}^{\infty} \gamma_j g_j = x$, that is, $\lim_{n\to\infty} \|\sum_{j=1}^{n} \gamma_j g_j - x\| = 0$.*

A Schauder basis $(g_j)_{j=1}^{\infty}$ is said to be bounded if $0 < \inf_{j\geq 1} \|g_j\| \leq \inf_{j\geq 1} \|g_j\| < \infty$.

It is, of course, possible to replace the index set $\mathbb{N}$ by some other countable set $\mathbb{I}$, but then one must be more careful with the convergence and provide an explicit bijection $\mathbb{N} \to \mathbb{I}$ or equivalently, give a linear ordering of the set $(g_j)_{j\in\mathbb{I}}$.

It is clear that if there is a Schauder basis in a Banach space, then this space must be separable, i.e., there is a countable set that is dense in the space.

If the dimension of the space is finite, then a set is a basis if and only if it is linearly independent and spans the space. To see that this is not sufficient in the infinite dimensional case consider the set $(\bullet^j)_{j=0}^{\infty}$. This set of functions is linearly independent and by Weierstrass approximation theorem, it is also dense in $C([0,1];\mathbb{C})$. However, it is not a basis, because if it were, then every continuous function could be written in the form

$$f = \sum_{j=0}^{\infty} \gamma_j \bullet^j, \tag{3.1}$$

where the series converges uniformly on $[0,1]$. But then f is an analytic function and this is clearly not always the case.

The following result describes what kind of "independence" is needed in the infinite dimensional case and it also turns out to be very useful when one wants to check whether a given sequence is a basis or not.

Theorem 3.2. *Let V be a Banach space with norm $\|\cdot\|$. The sequence $(g_j)_{j=1}^{\infty} \subset$ is a Schauder basis for V if and only if*

(i) $g_j \neq 0$ *for every* $j \geq 1$,

(ii) $\overline{\operatorname{span}\{\sum_{j=1}^{\infty}\}} = V$,

(iii) *there exists a positive number K such that*

$$\left\|\sum_{j=1}^{n} \gamma_j g_j\right\| \leq K \left\|\sum_{j=1}^{n+k} \gamma_j g_j\right\|,$$

for all positive integers n and k and for all scalars γ_j.

Proof. Assume that $(g_j)_{j\in\mathbb{N}}$ is a Schauder basis for V. Properties (i) and (ii) follow almost immediately from the definition of a basis so it remains to establish (iii). We define $|||\cdot|||$ by

$$|||x||| = \sup_{k\geq 1}\left\|\sum_{j=1}^{k} \gamma_j g_j\right\| \quad \text{if} \quad x = \sum_{j=1}^{\infty} \gamma_j g_j. \tag{3.2}$$

It is easy to check that $|||\cdot|||$ is a norm in V, but we must also prove that $V_{|||\cdot|||}$ (the space V with the norm $|||\cdot|||$) is complete. Let us therefore assume that $(x_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $V_{|||\cdot|||}$. Since $(g_j)_{j\in\mathbb{N}}$ is a basis, it follows that for each $m \geq 1$ we have $x_m = \sum_{j=1}^{\infty} \gamma_j(x_m) g_j$. Now we have for $p > q$

$$\begin{aligned} &\left\|\sum_{j=q+1}^{p} \big(\gamma_j(x_m) - \gamma_j(x_n)\big) g_j\right\| \\ &\quad\leq \left\|\sum_{j=1}^{p} \big(\gamma_j(x_m) - \gamma_j(x_n)\big) g_j\right\| + \left\|\sum_{j=1}^{q} \big(\gamma_j(x_m) - \gamma_j(x_n)\big) g_j\right\| \\ &\quad\leq 2|||x_m - x_n|||. \end{aligned} \tag{3.3}$$

By taking $p = q+1$ we see that $(\gamma_p(x_m) g_p)_{m\in\mathbb{Z}}$ is a Cauchy sequence (in $V_{\|\cdot\|}$) as well, and therefore it converges towards some element $\eta_p g_p$.

Let $\epsilon > 0$ be arbitrary. There exists an integer m_ϵ such that $|||x_m - x_n||| < \epsilon/4$ when $m, n \geq m_\epsilon$. If we let $n \to \infty$ in (3.3), then we get for ever $m \geq m_\epsilon$

$$\left\|\sum_{j=q+1}^{p} \gamma_j(x_m) g_j - \sum_{j=q+1}^{p} \eta_j g_j\right\| \leq \tfrac{\epsilon}{2}. \tag{3.4}$$

Now $\sum_{j=1}^{p}\gamma_j(x_{m_\epsilon})g_j$ converges towards x_{m_ϵ} and hence there is a number q_ϵ such that $\|\sum_{j=q+1}^{p}\gamma_j(x_{m_\epsilon})g_j\| < \epsilon/2$ when $q_\epsilon \le q < p$. But then it follows from (3.4) with $m = m_\epsilon$ that

$$\left\|\sum_{j=q+1}^{p}\eta_j g_j\right\| < \epsilon, \tag{3.5}$$

and the completeness of $V_{\|\cdot\|}$ implies that the series $\sum_{j=1}^{\infty}\eta_j g_j$ converges toward some element x of V. By the uniqueness of the coefficients $\gamma_j(x)$, we know that $\eta_j = \gamma_j(x)$ for all $j \in \mathbb{N}$. Taking $q = 0$ in (3.4) we have

$$|||x_m - x||| = \sup_{p\ge 1}\left\|\sum_{j=1}^{p}\big(\gamma_j(x_m)g_j - \sum_{j=1}^{p}\gamma_j(x)g_j\big)\right\| < \epsilon, \tag{3.6}$$

when $m \ge m_\epsilon$ and thus we see that $V_{|||\cdot|||}$ is complete.

Since $\|\bullet\| \le |||\bullet|||$, the identity mapping $T : V_{|||\cdot|||} \to V_{\|\cdot\|}$ is continuous, and since $V_{|||\cdot|||}$ is complete, it follows from the inverse mapping theorem that T^{-1} is continuous too, and this is exactly what (iii) says.

Next we have to prove that conditions (i)–(iii) are sufficient for $(g_j)_{j\in\mathbb{N}}$ to be a Schauder basis. For each $k \ge 1$, let g_k^* be the linear functional defined on $\operatorname{span}(g_j)_{j\in\mathbb{Z}}$ by letting $\langle g_k^*, x\rangle = \gamma_k$ if $x = \sum_{j=1}^{\infty}\gamma_j g_j$ with only finitely many nonzero terms. Moreover, we then have for these elements x

$$|\langle g_k^*, x\rangle| = |\gamma_k| = \frac{\left\|\sum_{j=1}^{k}\gamma_j g_j - \sum_{j=1}^{k-1}\gamma_j g_j\right\|}{\|g_k\|} \le \frac{2K\|x\|}{\|g_k\|},$$

so that g_k^* is continuos with norm

$$\|g_k^*\| \le \frac{2K}{\|g_k\|}. \tag{3.7}$$

By (ii) it is possible to extend g_k^* by continuity to all of V. One consequence of this result is that if $x = \sum_{j=1}^{\infty}\gamma_j g_j$, then the coefficients are uniquely determined by $\gamma_j = g_j^*(x)$.

Next we consider the element $h_n = x - \sum_{j=1}^{n}\left\langle g_j^*, x\right\rangle g_j$, where $n \ge 1$. By (ii) h_n belongs to the closure of $\operatorname{span}(g_j)_{j\ge 1}$, but since $g_j^*(h_n) = 0$ for every integer j between 1 and n, it follows that h_n actually belongs to $\operatorname{span}(g_j)_{j\ge n+1}$.

Let $\epsilon > 0$ be arbitrary. Invoking (ii) once more, we see that there exist an integer N and numbers γ_j, $j \ge 1$ with $\gamma_j = 0$ when $j > N$ such that $\|\sum_{j=1}^{N}\gamma_j g_j - x\| < \epsilon/(1+K)$ Then

$$\left\|\sum_{j=1}^{n}\big(\langle g_j^*, x\rangle - c_j\big)g_j + h_n\right\| < \tfrac{\epsilon}{1+K}, \quad n \ge N. \tag{3.8}$$

Since $h_n \in \overline{\operatorname{span}(g_j)_{j\geq n+1}}$ we can invoke (iii) to conclude that

$$\left\| \sum_{j=1}^{n} \bigl(\langle g_j^*, x\rangle - \gamma_j\bigr) g_j \right\| < \tfrac{\epsilon K}{1+K}, \quad n \geq N, \tag{3.9}$$

and it follows from the triangle inequality that

$$\|h_n\| < \epsilon, \quad n \geq N. \tag{3.10}$$

This completes the proof. □

2. Unconditional bases

A Schauder basis in a Banach space is said to be unconditional if, whenever the sum $\sum_{j=1}^{\infty} \gamma_j g_j$ converges, it actually converges unconditionally, i.e., if every permutation of the series converges. An immediate property of unconditional bases is that there are no problems with the summation if the index set is an arbitrary countable set $\mathbb{I}$, instead of $\mathbb{N}$.

Recall that if a series converges absolutely in a Banach space, then it converges unconditionally. In finite dimensional spaces the converse also holds but this is no longer the case if the dimension is infinite.

Next we give a useful characterization of unconditional bases that is analoguous to Theorem 3.2.

Theorem 3.3. *Let V be a Banach space with norm $\|\cdot\|$. The set $(g_j)_{j\in\mathbb{I}}$ is an unconditional basis for V if and only if*

(i) *$g_j \neq 0$ for every $j \in \mathbb{I}$,*

(ii) *$\overline{\operatorname{span}\{g_j\}_{j\in\mathbb{I}}} = V$,*

(iii) *there exists a positive integer K such that*

$$\left\| \sum_{j\in A} \gamma_j g_j \right\| \leq K \left\| \sum_{j\in A\cup B} \gamma_j g_j \right\|, \tag{3.11}$$

for all finite subsets A and B of $\mathbb{I}$ and all scalars γ_j.

3. Orthonormal and Riesz bases in Hilbert spaces

First we consider a Hilbert space and give some equivalent conditions for a sequence to be an orthonormal basis. Usually the space H is separable and the sequence countable, but this is not necessary.

Theorem 3.4. *Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and assume that $e_n \in H$ for all $n \in \mathbb{I}$. Then the following properties are equivalent (and if they hold the sequence $(e_n)_{n\in\mathbb{I}}$ is said to be an orthonormal basis for H):*

(i) $\overline{\operatorname{span}\{e_n\}_{n\in\mathbb{I}}} = H$ *and*

$$\langle e_n, e_m\rangle = \begin{cases} 1, & \text{if } n = m \in \mathbb{I}, \\ 0, & \text{if } n \neq m,\ n, m \in \mathbb{I}. \end{cases}$$

(ii) $\overline{\operatorname{span}\{e_n\}_{n\in\mathbb{I}}} = H$ *and*

$$\sum_{n=A} |c_n|^2 = \left\| \sum_{n\in A} c_n e_n \right\|^2,$$

for all numbers c_n, $n \in A$, *where* A *is a finite subset of* $\mathbb{I}$.

(iii) $\|e_n\| = 1$, $n \in \mathbb{I}$ *and*

$$\sum_{n\in\mathbb{I}} |\langle f, e_n\rangle|^2 = \|f\|^2, \quad f \in H.$$

Next we consider so called Riesz bases, but note that there are other ways of characterizing such bases than the ones given below.

Theorem 3.5. *Let* H *be a Hilbert space with inner product* $\langle\cdot,\cdot\rangle$ *and let* $f_n \in H$ *for all* $n \in \mathbb{I}$. *Then the following properties are equivalent (and if they hold, the sequence* $(f_n)_{n\in\mathbb{I}}$ *is said to be a Riesz basis):*

(i) *There is an orthonormal basis* $(e_n)_{n\in\mathbb{I}}$ *of* H *and a bounded linear operator* $T : H \to H$ *with bounded inverse such that* $f_n = Te_n$ *for each* $n \in \mathbb{I}$.

(ii) $(f_n)_{n\in\mathbb{I}}$ *is an unconditional basis for* H *and there are positive constants* α *and* β *such that* $0 < \alpha \leq \|f_n\| \leq \beta < \infty$ *for all* $n \in \mathbb{I}$.

(iii) $\overline{\operatorname{span}\{f_n\}_{n\in\mathbb{I}}} = H$ *and there are positive constants* a *and* b *such that and*

$$a \sum_{n\in A} |c_n|^2 \leq \left\| \sum_{n\in A} c_n e_n \right\|^2 \leq b \sum_{n\in A} |c_n|^2,$$

for all numbers c_n, $n \in A$ *where* A *is a finite subset of* $\mathbb{I}$.

(iv) $\overline{\operatorname{span}\{f_n\}_{n\in\mathbb{I}}} = H$ *and there are positive constants* a *and* B *such that*

$$a \sum_{n=1}^{k} |c_n|^2 \leq \|\sum_{n=1}^{k} c_n e_n\|^2,$$

for all numbers $c_1, \dots, c_k$, $k \geq 1$, *and*

$$\sum_{n\in\mathbb{I}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \quad f \in H.$$

(v) $\overline{\operatorname{span}\{f_n\}_{n\in\mathbb{I}}} = H$ *and there is a sequence* $(g_n)_{n\in\mathbb{I}}$ *such that* $\overline{\operatorname{span}\{g_n\}_{n\in\mathbb{I}}} = H$ *and for all* $m,\ n \geq 1$ *we have* $\langle f_n, g_m\rangle = 0$ *if* $n \neq m$ *and* $\langle f_n, g_n\rangle = 1$, *and there is a constant* B *such that*

$$\left.\begin{aligned} &\sum_{n\in\mathbb{I}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \\ &\sum_{n\in\mathbb{I}} |\langle f, g_n\rangle|^2 \leq B\|f\|^2, \end{aligned}\right. \qquad f \in H.$$

(vi) *There is a sequence* $(g_n)_{n\in\mathbb{I}}$ *such that for all* $m,\ n \geq 1$ *we have* $\langle f_n, g_m\rangle = 0$ *if* $n \neq m$ *and* $\langle f_n, g_n\rangle = 1$, *and there are constants* $0 < A \leq B < \infty$ *such that*

$$\left.\begin{aligned} A\|f\|^2 \leq &\sum_{n\in\mathbb{I}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \\ A\|f\|^2 \leq &\sum_{n\in\mathbb{I}} |\langle f, g_n\rangle|^2 \leq B\|f\|^2, \end{aligned}\right. \qquad f \in H.$$

Proof. (ii)$\Rightarrow$(iii): It is part of the definition of an unconditional basis that $\overline{\operatorname{span}\{f_n\}_{n\in\mathbb{I}}} = H$.

Suppose next that $A \subset \mathbb{I}$ is a finite set with $\#A$ number of elements. Let θ be a function: $A \to \{-1, 1\}$ and let M_θ be the function on $\operatorname{span}\{f_n\}_{n\in A}$ defined by $M_\theta(\sum_{n\in A} c_n f_n) = \sum_{n\in A} \theta(n) c_n f_n$. By Theorem 3.3 there is a constant K such that

$$\Bigg\| \sum_{\substack{n\in A \\ \theta(n)=1}} c_n f_n \Bigg\| \leq K \Bigg\| \sum_{n\in A} c_n f_n \Bigg\|, \quad \text{and} \quad \Bigg\| \sum_{\substack{n\in A \\ \theta(n)=-1}} c_n f_n \Bigg\| \leq K \Bigg\| \sum_{n\in A} c_n f_n \Bigg\|.$$

From these inequalities we can conclude that

$$\|M_\theta\| \leq 2K. \tag{3.12}$$

Next, let Θ_A be the set of functions $\theta : A \to \{-1, 1\}$. Clearly there are $2^{\#A}$ elements in this set. A straightforward calculation shows that

$$2^{-\#A} \sum_{\theta\in\Theta_A} \left\| M_\theta \left(\sum_{n\in A} c_n f_n \right) \right\|^2 = 2^{-\#A} \sum_{\theta\in\Theta_A} \left\| \sum_{n\in A} \theta(n) c_n f_n \right\|^2 = 2^{-\#A} \sum_{\theta\in\Theta_A} \Bigg(\sum_{n\in A} |c_n|^2 \|f_n\|^2 + \sum_{\substack{n,m\in A \\ n\neq m}} \theta($$

$$= \sum_{n\in A} |c_n|^2 \|f_n\|^2 + 2^{-\#A} \sum_{\substack{n,m\in A \\ n\neq m}} \langle c_n f_n, c_m f_m\rangle \sum_{\theta\in\Theta_A} \theta(n)\theta(m) = \sum_{n\in A} |c_n|^2 \|f_n\|^2.$$

Using (3.12) we see from the previous equality that

$$\sum_{n\in A} |c_n|^2 \|f_n\|^2 \le 4K^2 \left\| \sum_{n\in A} c_n f_n \right\|^2,$$

and because $M_\theta M_\theta = I$ that

$$\begin{aligned} \left\| \sum_{n\in A} c_n f_n \right\|^2 &= 2^{-\#A} \sum_{\theta\in\Theta_A} \left\| M_\theta M_\theta \left(\sum_{n\in A} c_n f_n \right) \right\|^2 \\ &\le 4K^2 2^{-\#A} \sum_{\theta\in\Theta_A} \left\| M_\theta \left(\sum_{n\in A} c_n f_n \right) \right\|^2 = 4K^2 \sum_{n\in A} |c_n|^2 \|f_n\|^2. \end{aligned}$$

Thus we conclude from the assumption that the basis $(f_n)_{n\in\mathbb{I}}$ is bounded that

$$\frac{\alpha^2}{4K^2} \sum_{n\in A} |c_n|^2 \le \left\| \sum_{n\in A} c_n f_n \right\|^2 \le 4K^2\beta^2 \sum_{n\in A} |c_n|^2,$$

and this is exactly what we had to prove.

(i)⇒(ii): Let $f \in H$ be arbitrary. Since $(e_n)_{n\in\mathbb{I}}$ is an orthonormal basis we have

$$T^{-1} f = \sum_{n\in\mathbb{I}} c_n e_n,$$

(where, of course, $c_n = \langle T^{-1} f, e_n \rangle$, $n \in I$). Since T is continuous and $Te_n = f_n$ for $n \in \mathbb{I}$, it follows that

$$f = \sum_{n\in\mathbb{I}} c_n f_n.$$

The uniqueness of the coefficients follows from the uniqueness of the expansion in the orthonormal basis and thus we conclude that $(f_n)_{n\in\mathbb{I}}$ is an unconditional basis. Because $\|e_n\| = 1$ it follows that

$$\frac{1}{\|T^{-1}\|} \le \|f_n\| \le \|T\|, \quad n \in \mathbb{I}.$$

(i)⇒(iii): Since $f_n = Te_n$ for all n we have

$$\sum_{n\in A} c_n f_n = T\left(\sum_{n\in A} c_n e_n \right) \quad \text{and} \quad T^{-1} \sum_{n\in A} c_n f_n = \left(\sum_{n\in A} c_n e_n \right)$$

so that

$$\left\| \sum_{n\in A} c_n f_n \right\|^2 \le \|T\|^2 \left\| \sum_{n\in A} c_n e_n \right\|^2 = \|T\|^2 \sum_{k=1}^{n} |c_n|^2,$$

and

$$\left\|\sum_{n\in A} c_n f_n\right\|^2 \geq \|T^{-1}\|^{-2}\left\|\sum_{n\in A} c_n e_n\right\|^2 = \|T^{-1}\|^{-2}\sum_{k=1}^{n}|c_n|^2.$$

(iii)⇔(iv): Suppose (iii) holds. If c_n, $n = 1, \dots, k$ are arbitrary numbers we have

$$\left|\sum_{n\in A} c_n \langle f, f_n\rangle\right|^2 = \left|\left\langle f, \sum_{n\in A} \overline{c_n} f_n\right\rangle\right|^2 \leq \|f\|^2 \left\|\sum_{n\in A} \overline{c_n} f_n\right\|^2 \leq b\|f\|^2 \sum_{n\in A}|c_k|^2.$$

If we now choose $c_n = \overline{\langle f, f_n\rangle}$ and let $k \to \infty$, then we get the missing claim.

For the converse we let $f = \sum_{n\in A} c_n f_n$. Then

$$\|f\|^4 = |\langle f, f\rangle|^2 = \left|\sum_{n=1} k\overline{c_n}\langle f, f_n\rangle\right|^2$$
$$\leq \sum_{n\in A}|c_n|^2 \sum_{n\in A}|\langle f, f_n\rangle|^2 \leq B\|f\|^2 \sum_{n\in A}|c_n|^2.$$

When we divide by $\|f\|^2$ we get the desired result.

(iv)⇒(v): The first inequality implies that for each $m \geq 1$

$$\left\|\sum_{\substack{n=1\\ n\neq m}}^{k} c_n f_n - f_m\right\| \geq a > 0.$$

Thus $f_m \notin \overline{\operatorname{span}\{\, f_n \mid n \geq 1, n \neq m\,\}}$ and therefore there exists an element $g_m \in H$ such that $\langle f_n, g_m\rangle = 0$ if $n \neq m$ and 1 if $n = m$.

If $f = \sum_{n\in A} c_n f_n$ we must therefore have $c_n = \langle f, g_n\rangle$. Thus we have

$$\sum_{n\in\mathbb{I}}|\langle f, g_n\rangle|^2 \leq \frac{1}{a}\|f\|^2,$$

for f in a dense subset of H, and by continuity for all $f \in H$. In order to prove that $\overline{\operatorname{span}\{g_n\}_{n\in\mathbb{I}}} = H$ it suffices to recall that (iv) implies (iii) because then we can conclude that if for some $f \in H$ we have $\langle f, g_n\rangle = 0$ for all $n \geq 1$ then $f = 0$.

(v)⇒(i): Let $(e_n)_{n\in\mathbb{I}}$ be an arbitrary orthonormal basis for H. furthermore, Let $f = \sum_{n\in A} c_n f_n$ and $g = \sum_{n\in A} d_n g_n$. By the biorthogonality assumption we have $c_n = \langle f, g_n\rangle$ and $d_n = \langle g, f_n\rangle$. If we now define

$$Sf = \sum_{n\in A} c_n e_n,$$
$$Ug = \sum_{n=1}^{x} k d_n e_n,$$

then we conclude that

$$\|Sf\|^2 = \sum_{n\in A} |c_n|^2 = \sum_{n\in A} |\langle f, g_n\rangle|^2 \le B\|f\|^2.$$

A similar inequality can be derived for U so that we conclude, since S and U are densely defined that they can be extended to bounded continuous operators on H with norms at most $\sqrt{B}$. The biorthogonality combined with the continuous extension implies that

$$\langle Sf, Ug\rangle = \langle f, g\rangle, \quad f, g \in H.$$

Thus we conclude that

$$\|f\|^2 = \langle f, g\rangle = \langle Sf, Uf\rangle \le \|Sf\|\|Uf\| \le \|Sf\|\sqrt{B}\|f\|.$$

Since the range of S is dense in H we conclude that S has a bounded inverse and the proof is completed.

(v)⇔(vi): First assume that (v) holds. Since we know that (v) is equivalent to (i) there is an operator T such that $(T^{-1}f_n)_{n\in\mathbb{I}}$ is an orthonormal basis. Then

$$\sum_{n\in\mathbb{I}} |\langle f, f_n\rangle|^2 = \sum_{n\in\mathbb{I}} \left|\left\langle f, TT^{-1}f_n\right\rangle\right|^2$$
$$= \sum_{n\in\mathbb{I}} \left|\left\langle T^*f, T^{-1}f_n\right\rangle\right|^2 = \|T^*f\|^2 \ge \frac{1}{\|(T^*)^{-1}\|^2}\|f\|^2.$$

Since $(g_n)_{n\in\mathbb{I}}$ satisfies the same assumptions as $(f_n)_{n\in\mathbb{I}}$ we get the second conclusion as well.

Suppose next that (vi) holds. Then we have only to establish the fact that $\overline{\text{span}\{f_n\}_{n\in\mathbb{I}}} = H$ and $\overline{\text{span}\{g_n\}_{n\in\mathbb{I}}} = H$ and these claims follow directly because by (vi) there cannot be a nonzero vector orthogonal to all vectors f_n or to all vectors g_n. □

4. Frames

If the first condition in Theorem 3.5.(iv) holds, then we have a Riesz-Fischer sequence and if the second one holds then we have a Bessel sequence. However, here we shall consider the case where we require the first inequality in 3.5.(vi) to hold.

Definition 3.6. *Let H be a separable Hilbert space. A sequence $(f_n)_{n\in\mathbb{I}}$ of elements in H is a frame if there are positive constants A and B (the bounds for the frame) such that*

$$A\|f\|^2 \le \sum_{n\in\mathbb{I}} |\langle f, f_n\rangle|^2 \le B\|f\|^2, \quad f \in H.$$

Theorem 3.7. *If $(f_n)_{n\in\mathbb{I}}$ is a frame then the formula*

$$Tf = \sum_{n\in\mathbb{I}} \langle f, f_n\rangle f_n. \tag{3.13}$$

defines a bounded, selfadjoint, invertible, linear operator with $\|T\| \le B$ and $\|T^{-1}\| \le A^{-1}$. Moreover, if $f \in H$, then

$$f = \sum_{n\in\mathbb{I}} a_n f_n \quad \text{where} \quad a_n = \left\langle T^{-1}f, f_n\right\rangle = \left\langle f, T^{-1}f_n\right\rangle, \quad n \ge 1,$$

and if $f = \sum_{n\in\mathbb{I}} b_n f_n$, then

$$\sum_{n\in\mathbb{I}}|b_n|^2 = \sum_{n\in\mathbb{I}}|a_n|^2 + \sum_{n\in\mathbb{I}}\left|a_n - b_n\right|^2 \ge \sum_{n\in\mathbb{I}}|a_n|^2. \tag{3.14}$$

Proof. First we have to show that T is well defined. Let $\mathbb{J}$ be a finite subset of $\mathbb{I}$ and

$$T_{\mathbb{J}}f = \sum_{n\in\mathbb{J}} \langle f, f_n\rangle f_n.$$

Observe that

$$\begin{aligned}\|T_{\mathbb{J}}f\|^4 = |\langle T_{\mathbb{J}}f, T_{\mathbb{J}}f\rangle|^2 &= \left|\sum_{n\in\mathbb{J}} \langle f, f_n\rangle \langle f_n, T_{\mathbb{J}}f\rangle\right|^2 \\ &\le \sum_{n\in\mathbb{J}}|\langle f, f_n\rangle|^2 \sum_{n\in\mathbb{J}}|\langle f_n, T_{\mathbb{J}}f\rangle|^2 \le \begin{cases} B^2\|f\|^2\|T_{\mathbb{J}}f\|^2, \\ B\sum_{n\in\mathbb{J}}|\langle f, f_n\rangle|^2\|T_{\mathcal{J}}f\|^2. \end{cases}\end{aligned}$$

Thus we conclude that

$$\|T_{\mathbb{J}}f\| \le B\|f\|,$$

and

$$\|T_{\mathbb{J}}f\|^2 \le B\sum_{n\in\mathbb{J}}|\langle f, f_n\rangle|^2$$

From this we conclude that the sum $\sum_{n\in\mathbb{I}} \langle f, f_n\rangle f_n$ converges to an element Tf where T is a linear operator satisfying

$$\|T\| \le B.$$

Next we observe that

$$\langle Tf, f\rangle = \sum_{n\in\mathbb{I}}|\langle f, f_n\rangle|^2 \ge A\|f\|^2.$$

From this we first conclude that $\|Tf\| \ge A\|f\|$ which implies that the range of T is closed. If this range is not H there is a nonzero vector $h \in H$ orthogonal to it, but this is impossible because $\langle Th, h\rangle \ge A\|h\|^2 > 0$. Thus we conclude that T is invertible.

Next we show that T is self-adjoint. Let f and $g \in H$ be arbitrary. Then

$$\langle Tf, g\rangle = \sum_{n\in\mathbb{I}} \langle f, f_n\rangle \langle f_n, g\rangle = \sum_{n\in\mathbb{I}} \langle f, f_n\rangle \overline{\langle f_n, g\rangle}$$
$$= \left\langle f, \sum_{n\in\mathbb{I}} \langle f_n, g\rangle f_n \right\rangle = \langle f, Tg\rangle .$$

By the definition of T we have

$$f = T(T^{-1}f) = \sum_{n=1} \left\langle T^{-1}f, f_n\right\rangle f_n = \sum_{n=1} \left\langle f, T^{-1}f_n, f\right\rangle_n .$$

Suppose now that $f = \sum_{n\in\mathbb{I}} b_b f_n$. If $\sum_{n\in\mathbb{I}} |b_n|^2 = \infty$ there is nothing to prove, so we assume that the sum is finite. Now we have

$$\sum_{n\in\mathbb{I}} (b_n - a_n)\overline{a_n} = \sum_{n\in\mathbb{I}} (b_n - a_n) \left\langle f_n, T^{-1}f\right\rangle = \left\langle \sum_{n\in I} (b_n - a_n) f_n, T^{-1}f \right\rangle$$
$$= \left\langle \sum_{n\in\mathbb{I}} b_n f_n - \sum_{n\in\mathbb{I}} a_n f_n, T^{-1}f \right\rangle = \left\langle f - f, T^{-1}f\right\rangle = 0,$$

which means that $(b_n - a_n)_{n\in\mathbb{I}} \perp (a_n)_{n\in\mathbb{I}}$ and using this fact we get (3.14).

□

Theorem 3.8. *Let H be a separable Hilbert space and let $(f_n)_{n\in\mathbb{I}}$ be a frame in H. Let $g_n = T^{-1}f_n$ where T is the operator $Tf = \sum_{n\in\mathbb{I}} \langle f, f_n\rangle f_n$. Then either $(f_n)_{n\in\mathbb{I}}$ is a Riesz basis for H (with $\langle f_n, g_m\rangle = 0$ if $n \neq m$ and 1 if $n = m$) or there is a number $k \geq 1$ such that $(f_n)_{\substack{n=1\\n\neq k}}^{\infty}$ is a frame.*

Proof. If for all m and $n \geq 1$ we have

$$\langle f_n, g_m\rangle = \begin{cases} 0, & \text{if } n \neq m, \\ 1, & \text{if } n = m, \end{cases}$$

then $(f_n)_{n\in\mathbb{I}}$ is a Riesz basis by Theorem 3.5.(vi).

Suppose that for some $k \geq 1$ either $\langle f_k, g_k\rangle \neq 1$ or $\langle f_k, g_m\rangle \neq 0$ for some $m \neq k$. Since $(f_n)_{n\in\mathbb{I}}$ is a frame we have write

$$f_k = \sum_{n\in\mathbb{I}} \langle f_k, g_n\rangle f_n .$$

If now $\langle f_k, g_k\rangle = 1$ then we have

$$0 = \sum_{\substack{n=1\\n\neq k}}^{\infty} \langle f_k, g_n\rangle f_n$$

On the other hand we have

$$0 = \sum_{n \in \mathbb{I}} 0\, f_n,$$

and by Theorem 3.7 we must therefore have

$$\langle f_k, g_n \rangle = 0, \quad n \neq k.$$

Thus we may assume that $a_k \stackrel{\text{def}}{=} \langle f_k, g_k \rangle \neq 1$. Then we have

$$f_k = \frac{1}{1 - a_k} \sum_{\substack{n=1 \\ n \neq k}}^{\infty} \langle f_k, g_n \rangle\, f_n,$$

and in particular

$$|\langle f, f_k \rangle|^2 = \frac{1}{|1 - a_k|^2} \left| \sum_{\substack{n=1 \\ n \neq k}}^{\infty} \overline{\langle f_k, g_n \rangle}\, \langle f, f_n \rangle \right|$$

$$\leq \frac{1}{|1 - a_k|^2} \sum_{\substack{n=1 \\ n \neq k}}^{\infty} |\langle f_k, g_n \rangle|^2 \sum_{\substack{n=1 \\ n \neq k}}^{\infty} |\langle f, f_n \rangle|^2.$$

Thus we conclude that

$$\sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq C \|f\|^2,$$

where $C = 1 + \frac{1}{|1 - a_k|^2} \sum_{\substack{n=1 \\ n \neq k}}^{\infty} |\langle f_k, g_n \rangle|^2$. It follows that

$$\frac{A}{C} \|f\|^2 \leq \sum_{\substack{n=1 \\ n \neq k}}^{\infty} |\langle f, f_n \rangle|^2 \leq B \|f\|^2,$$

and we conclude that $(f_n)_{\substack{n=1 \\ n \neq k}}^{\infty}$ is a frame. This completes the proof. □