TKK, Institute of mathematics Gripenberg
Mat-1.3422 Wavelet theory
Exercise 2
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1. Let $a \in \ell^1(\mathbb{Z})$ and define the sequence b by $b(\underline{n}) = a(2\underline{n})$. Express the Fourier transfrom $\hat{b}$ of the sequence b wiht the aid of the Fourier transform $\hat{a}$ of the sequence a.

2. Let $\alpha \in \ell^2(\mathbb{Z})$ (and you may assume that there are only finitle many non-zero elements in the sequence). Show that

$$|\hat{\alpha}(\underline{\omega})|^2 + |\hat{\alpha}(\underline{\omega} + \tfrac{1}{2})|^2 \overset{\text{a.e.}}{=} 1,$$

if and only if

$$2\sum_{k\in\mathbb{Z}} \alpha(k)\overline{\alpha(k+2n)} = \delta_{0,n}, \quad n \in \mathbb{Z},$$

by first calculating the Fourier transform of the sequence $(\sum_{k\in\mathbb{Z}} \alpha(k)\overline{\alpha(k-n)})_{n\in\mathbb{Z}}$.

3. What can be said about the following argument: If $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ is such that $\int_{\mathbb{R}} \psi(t)\,\mathrm{d}t = 0$ then it is also true that $\int_{\mathbb{R}} \psi\left(2^{-m}t - k\right)\,\mathrm{d}t = 0$ for all integres m and k and then it is not possible to write for example

$$\mathrm{e}^{-\underline{t}^2} = \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{m,k}\psi\left(2^{-m}\underline{t} - k\right),$$

where the series converges in $L^2(\mathbb{R})$ because $\int_{\mathbb{R}} \mathrm{e}^{-t^2}\,\mathrm{d}t > 0$.

4. Let $\alpha \in l^1(\mathbb{Z})$. Define the operators T_α and $S_\alpha : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ as follows:

$$(T_\alpha c)(k) = \sum_{j\in\mathbb{Z}} \overline{\alpha(j-2k)}c(j) \quad \text{and} \quad (S_\alpha c)(k) = 2\sum_{j\in\mathbb{Z}} \alpha(k-2j)c(j).$$

What are the operators T_α^* and S_α^* (defined by the requirements that $\langle T_\alpha c, d\rangle = \langle c, T_\alpha^* d\rangle$ and $\langle S_\alpha c, d\rangle = \langle c, S_\alpha^* d\rangle$)? Under which conditions is it true that $T_\alpha T_\alpha^* = \frac{1}{2}I$?

5. Let $(\{V_m\}_{m\in\mathbb{Z}}, \varphi)$ be a multiresolution for $L^2(\mathbb{R};\mathbb{C})$ and let P_m denote the orthogonal projection onto V_m. If $f \in V_m$ is given in the form

$$f = \sum_{k\in\mathbb{Z}} C_m(k)2^{-\frac{m}{2}}\varphi(2^{-m}\bullet - k),$$

where $C_m \in \ell^2(\mathbb{Z};\mathbb{C})$, then

$$P_{V_{m+1}}f = \sum_{k\in\mathbb{Z}} C_{m+1}(k)2^{\frac{-m-1}{2}}\varphi(2^{-m-1}\bullet - k).$$

Determine the coefficients $C_{m+1}(k)$ in terms of the coefficients $C_m(k)$.

Do the same for the projection onto the space W_{m+1}.