3 Topology (and metric), basics

The reader should know metric spaces; topological spaces are their generalization, which we soon introduce. Feel free to draw some clarifying schematic pictures on the margins!

Metric space. A function $d : X \times X \to [0, \infty[$ is called a *metric* on the set X if for every $x, y, z \in X$ we have

- $d(x, y) = 0 \Leftrightarrow x = y$;
- $d(x, y) = d(y, x)$;
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Then (X, d) (or simply X when d is evident) is called a *metric space*. Sometimes a metric is called a *distance function*.

Topological space. A family of sets $\tau \subset \mathcal{P}(X)$ is called a *topology* on the set X if

1. $\emptyset, X \in \tau$;
2. $\mathcal{U} \subset \tau \Rightarrow \bigcup \mathcal{U} \in \tau$;
3. $U, V \in \tau \Rightarrow U \cap V \in \tau$.

Then (X, τ) (or simply X when τ is evident) is called a *topological space*. The sets $U \in \tau$ are called *open sets*, and their complements $X \setminus U$ are *closed sets*.

Thus in a topological space, the empty set and the whole space are always open, **any** union of open sets is open, and an intersection of **finitely** many open sets is open. Equivalently, the whole space and the empty set are always closed, **any** intersection of closed sets is closed, and a union of **finitely** many closed sets is closed.

Metric topology. Let (X, d) be a metric space. We say that the *open ball of radius $r > 0$ centered at $x \in X$* is

$$B_d(x, r) := \{y \in X \mid d(x, y) < r\}.$$

The *metric topology* τ_d of (X, d) is given by

$$U \in \tau_d \overset{\text{definition}}{\Leftrightarrow} \forall x \in U \ \exists r > 0 : \ B_d(x, r) \subset U.$$

A topological space (X, τ) is called *metrizable* if there is a metric d on X such that $\tau = \tau_d$.

Non-metrizable spaces. There are plenty of non-metrizable topological spaces, the easiest example being X with more than one point and with $\tau = \{\emptyset, X\}$. If X is an infinite-dimensional Banach space then the weak*-topology of $X' := \mathcal{L}(X, \mathbb{C})$ is not metrizable. The distribution spaces $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ are non-metrizable topological spaces. We shall later prove that for the compact Hausdorff spaces metrizability is equivalent to the existence of a countable base.

Base. Let (X, τ) be a topological space. A family $\mathcal{B} \subset \tau$ of open sets is called a *base* (or *basis*) for the topology τ if any open set is a union of some members of $\mathcal{B}$, i.e.

$$\forall U \in \tau \ \exists \mathcal{B}' \subset \ \mathcal{B} : \ U = \bigcup \mathcal{B}'.$$

Examples. Trivially a topology τ is a base for itself ($\forall U \in \tau : U = \bigcup \{U\}$). If (X, d) is a metric space then

$$\mathcal{B} := \{B_d(x, r) \mid x \in X, \ r > 0\}$$

constitutes a base for τ_d.

Neighborhoods. Let (X, τ) be a topological space. A *neighborhood of* $x \in X$ is any open set $U \subset X$ containing x. The family of neighborhoods of $x \in X$ is denoted by

$$\mathcal{V}_\tau(x) := \{U \in \tau \mid x \in U\}$$

(or simply $\mathcal{V}(x)$, when τ is evident).

The natural mappings (or the morphisms) between topological spaces are **continuous mappings**.

Continuity at a point. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A mapping $f : X \to Y$ is *continuous at* $x \in X$ if

$$\forall V \in \mathcal{V}_{\tau_Y}(f(x)) \ \exists U \in \mathcal{V}_{\tau_X}(x) : \ f(U) \subset V.$$

Exercise. Let (X, d_X) and (Y, d_Y) be metric spaces. A mapping $f : X \to Y$ is continuous at $x \in X$ if and only if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in X :\ d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$$

if and only if

$$d_X(x_n, x) \to_{n\to\infty} 0 \Rightarrow d_Y(f(x_n), f(x)) \to_{n\to\infty} 0$$

for every sequence $(x_n)_{n=1}^{\infty} \subset X$ (that is, $x_n \to x \Rightarrow f(x_n) \to f(x)$).

Continuity. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A mapping $f : X \to Y$ is *continuous*, denoted by $f \in C(X, Y)$, if

$$\forall V \in \tau_Y : f^{-1}(V) \in \tau_X,$$

where $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$; i.e. f is continuous if preimages of open sets are open (equivalently, preimages of closed sets are closed). In the sequel, we briefly write

$$C(X) := C(X, \mathbb{C}),$$

where $\mathbb{C}$ has the metric topology with the usual metric $(\lambda, \mu) \mapsto |\lambda - \mu|$.

Proposition. *Let (X, τ_X) and (Y, τ_Y) be topological spaces. A mapping $f : X \to Y$ is continuous at every $x \in X$ if and only if it is continuous.*

Proof. Suppose $f : X \to Y$ is continuous, $x \in X$, and $V \in \mathcal{V}_{\tau_Y}(f(x))$. Then $U := f^{-1}(V)$ is open, $x \in U$, and $f(U) = V$, implying the continuity at $x \in X$.

Conversely, suppose $f : X \to Y$ is continuous at every $x \in X$, and let $V \subset Y$ be open. Choose $U_x \in \mathcal{V}_{\tau_X}(x)$ such that $f(U_x) \subset V$ for every $x \in f^{-1}(V)$. Then

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$$

is open in X □

Exercise. Let X be a topological space. Show that $C(X)$ is an algebra.

Exercise. Prove that if $f : X \to Y$ and $g : Y \to Z$ are continuous then $g \circ f : X \to Z$ is continuous.

Topological equivalence: homeomorphism. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A mapping $f : X \to Y$ is called a *homeomorphism* if it is a bijection, $f \in C(X, Y)$ and $f^{-1} \in C(Y, X)$. Then X and Y are called *homeomorphic* or *topologically equivalent*, denoted by $X \cong Y$ or $f : X \cong Y$; more specifically, $f : (X, \tau_X) \cong (Y, \tau_Y)$.

Note that from the topology point of view, homeomorphic spaces can be considered equal.

Examples. Of course $(x \mapsto x) : (X, \tau) \cong (X, \tau)$. The reader may check that $(x \mapsto x/(1+|x|)) : \mathbb{R} \cong]-1, 1[$. Using algebraic topology, one can prove that $\mathbb{R}^m \cong \mathbb{R}^n$ if and only if $m = n$ (this is not trivial!).

Metric equivalence and isometries. Metrics d_1, d_2 on a set X are called *equivalent* if there exists $m < \infty$ such that

$$M^{-1}\ d_1(x, y) \leq d_2(x, y) \leq M\ d_1(x, y)$$

for every $x, y \in X$. An *isometry* between metric spaces (X, d_X) and (Y, d_Y) is a mapping $f : X \to Y$ satisfying $d_Y(f(x), f(y)) = d_X(x, y)$ for every $x, y \in X$; f is called an *isometric isomorphism* if it is a surjective isometry (hence a bijection with an isometric isomorphism as the inverse mapping).

Examples. Any isometric isomorphism is a homeomorphism. Clearly the unbounded $\mathbb{R}$ and the bounded $]-1, 1[$ are not isometrically isomorphic. An orthogonal linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is an isometric isomorphism, when $\mathbb{R}^n$ is endowed with the Euclidean norm. The forward shift operator on $\ell^p(\mathbb{Z})$ is an isometric isomorphism, but the forward shift operator on $\ell^p(\mathbb{N})$ is only a non-surjective isometry.

Hausdorff space. A topological space (X, τ) is a *Hausdorff space* if any two distinct points have some disjoint neighborhoods, i.e.

$$\forall x, y \in X\ \exists U \in \mathcal{V}(x)\ \exists V \in \mathcal{V}(y) :\ x \neq y \Rightarrow U \cap V = \emptyset.$$

Examples.

1. If τ_1 and τ_2 are topologies of X, $\tau_1 \subset \tau_2$, and (X, τ_1) is a Hausdorff space then (X, τ_2) is a Hausdorff space.

2. $(X, \mathcal{P}(X))$ is a Hausdorff space.

3. If X has more than one point and $\tau = \{\emptyset, X\}$ then (X, τ) is not Hausdorff.

4. Clearly any metric space (X, d) is a Hausdorff space; if $x, y \in X$, $x \neq y$, then $B_d(x, r) \cap B_d(y, r) = \emptyset$, when $r \leq d(x, y)/2$.

5. The distribution spaces $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ are non-metrizable Hausdorff spaces.

Exercise*. Let X be a Hausdorff space and $x \in X$. Then $\{x\} \subset X$ is a closed set.

Finite product topology. Let X, Y be topological spaces with bases $\mathcal{B}_X, \mathcal{B}_Y$, respectively. Then a base for the *product topology* of $X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$ is

$$\{U \times V \mid U \in \mathcal{B}_X,\ V \in \mathcal{B}_Y\}.$$

Exercise. Let X, Y be metrizable. Prove that $X \times Y$ is metrizable, and that

$$(x_n, y_n) \stackrel{X \times Y}{\to} (x, y) \quad \Leftrightarrow \quad x_n \stackrel{X}{\to} x \text{ and } y_n \stackrel{Y}{\to} y.$$