2 Algebras

Algebra. A vector space $\mathcal{A}$ over the field $\mathbb{C}$ is an *algebra* if there exists an element $\mathbb{I}_{\mathcal{A}} \in \mathcal{A} \setminus \{0\}$ and a mapping $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(x, y) \mapsto xy$, satisfying

$$\begin{aligned}
&x(yz) = (xy)z, \\
&x(y+z) = xy + xz, \quad (x+y)z = xz + yz, \\
&\lambda(xy) = (\lambda x)y = x(\lambda y), \\
&\mathbb{I}_{\mathcal{A}} x = x = x \mathbb{I}_{\mathcal{A}}
\end{aligned}$$

for every $x, y, z \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. We briefly write $xyz := x(yz)$. The element $\mathbb{I} := \mathbb{I}_{\mathcal{A}}$ is called the *unit of* $\mathcal{A}$, and an element $x \in \mathcal{A}$ is called *invertible* (with the unique *inverse* x^{-1}) if there exists $x^{-1} \in \mathcal{A}$ such that

$$x^{-1}x = \mathbb{I} = xx^{-1}.$$

If $xy = yx$ for every $x, y \in \mathcal{A}$ then $\mathcal{A}$ is called *commutative.*

Warnings: In some books the algebra axioms allow $\mathbb{I}_{\mathcal{A}}$ to be 0, but then the resulting algebra is simply $\{0\}$; we have omitted such a triviality. In some books the existence of a unit is omitted from the algebra axioms; what we have called an algebra is there called a *unital algebra.*

Examples of algebras.

1. $\mathbb{C}$ is the most important algebra. The operations are the usual ones for complex numbers, and the unit element is $\mathbb{I}_{\mathbb{C}} = 1 \in \mathbb{C}$. Clearly $\mathbb{C}$ is a commutative algebra.

2. The algebra $\mathcal{F}(X) := \{f \mid f : X \to \mathbb{C}\}$ of complex valued functions on a (finite or infinite) set X is endowed with the same algebra structure as in the example in "Informal introduction" section (pointwise operations). Function algebras are commutative, because $\mathbb{C}$ is commutative.

3. The algebra $L(V) := \{A : V \to V \mid A \text{ is linear}\}$ of linear operators on a vector space $V \neq \{0\}$ over $\mathbb{C}$ is endowed with the usual vector space structure and with the multiplication $(A, B) \mapsto AB$ (composition of operators); the unit element is $\mathbb{I}_{L(V)} = (v \mapsto v) : V \to V$, the identity operator on V. This algebra is non-commutative if V is at least two-dimensional.

Exercise. Let $\mathcal{A}$ be an algebra and $x, y \in \mathcal{A}$. Prove the following claims:
(a) If x, xy are invertible then y is invertible.
(b) If xy, yx are invertible then x, y are invertible.

Exercise. Give an example of an algebra $\mathcal{A}$ and elements $x, y \in \mathcal{A}$ such that $xy = \mathbb{I}_{\mathcal{A}} \neq yx$. Prove that then $(yx)^2 = yx \neq 0$. (Hint: Such an algebra is necessarily infinite-dimensional).

Spectrum. Let $\mathcal{A}$ be an algebra. The *spectrum* $\sigma(x)$ of an element $x \in \mathcal{A}$ is the set

$$\sigma_{\mathcal{A}}(x) = \sigma(x) = \{\lambda \in \mathbb{C} : \ \lambda\mathbb{I} - x \text{ is not invertible}\}.$$

Examples of invertibility and spectra.

1. An element $\lambda \in \mathbb{C}$ is invertible if and only if $\lambda \neq 0$; the inverse of an invertible λ is the usual $\lambda^{-1} = 1/\lambda$. Generally, $\sigma_{\mathbb{C}}(\lambda) = \{\lambda\}$.

2. An element $f \in \mathcal{F}(X)$ is invertible if and only if $f(x) \neq 0$ for every $x \in X$. The inverse of an invertible f is g with $g(x) = f(x)^{-1}$. Generally, $\sigma_{\mathcal{F}(X)}(f) = f(X) := \{f(x) \mid x \in X\}$.

3. An element $A \in L(V)$ is invertible if and only if it is a bijection (if and only if $0 \notin \sigma_{L(V)}(A)$).

Exercise. Let $\mathcal{A}$ be an algebra and $x, y \in \mathcal{A}$. Prove the following claims:
(a) $\mathbb{I} - yx$ is invertible if and only if $\mathbb{I} - xy$ is invertible.
(b) $\sigma(yx) \subset \sigma(xy) \cup \{0\}$.
(c) If x is invertible then $\sigma(xy) = \sigma(yx)$.

Ideals. Let $\mathcal{A}$ be an algebra. An *ideal* $\mathcal{J} \subset \mathcal{A}$ is a vector subspace $\mathcal{J} \neq \mathcal{A}$ satisfying

$$\forall x \in \mathcal{A} \ \forall y \in \mathcal{J} : \ xy, yx \in \mathcal{J},$$

i.e. $x\mathcal{J}, \mathcal{J}x \subset \mathcal{J}$ for every $x \in \mathcal{A}$. A *maximal ideal* is an ideal not contained in any other ideal.

Warning. In some books our ideals are called *proper ideals*, and there *ideal* is either a proper ideal or the whole algebra.

Remark. Let $\mathcal{J} \subset \mathcal{A}$ be an ideal. Because $x\mathbb{I} = x$ for every $x \in \mathcal{A}$, we notice that $\mathbb{I} \notin \mathcal{J}$. Therefore an invertible element $x \in \mathcal{A}$ cannot belong to an ideal (since $x^{-1}x = \mathbb{I} \notin \mathcal{J}$).

Examples of ideals. Intuitively, an ideal of an algebra is a subspace resembling a multiplicative zero; consider equations $x0 = 0 = 0x$.

1. Let $\mathcal{A}$ be an algebra. Then $\{0\} \subset \mathcal{A}$ is an ideal.

2. The only ideal of $\mathbb{C}$ is $\{0\} \subset \mathbb{C}$.

3. Let X be a set, and $\emptyset \neq S \subset X$. Now

   $$\mathcal{I}(S) := \{f \in \mathcal{F}(X) \mid \forall x \in S : \ f(x) = 0\}$$

   is an ideal of the function algebra $\mathcal{F}(X)$. If $x \in X$ then $\mathcal{I}(\{x\})$ is a maximal ideal of $\mathcal{F}(X)$, because it is of co-dimension 1 in $\mathcal{F}(X)$. Notice that $\mathcal{I}(S) \subset \mathcal{I}(\{x\})$ for every $x \in S$; an ideal may be contained in many different maximal ideals (cf. Krull's Theorem in the sequel).

4. Let X be an infinite-dimensional Banach space. The set

   $$\mathcal{LC}(X) := \{A \in \mathcal{L}(X) \mid A \text{ is compact}\}$$

   of compact linear operators $X \to X$ is an ideal of the algebra $\mathcal{L}(X)$ of bounded linear operators $X \to X$.

Theorem (W. Krull). *An ideal is contained in a maximal ideal.*

Proof. Let $\mathcal{J}$ be an ideal of an algebra $\mathcal{A}$. Let P be the set of those ideals of $\mathcal{A}$ that contain $\mathcal{J}$. The inclusion relation is the natural partial order on P; the **Hausdorff Maximal Principle** says that there is a maximal chain $C \subset P$. Let $\mathcal{M} := \bigcup C$. Clearly $\mathcal{J} \subset \mathcal{M}$. Let $\lambda \in \mathbb{C}$, $x, y \in \mathcal{M}$ and $z \in \mathcal{A}$. Then there exists $\mathcal{I} \in C$ such that $x, y \in \mathcal{I}$, so that

$$\lambda x \in \mathcal{I} \subset \mathcal{M}, \quad x + y \in \mathcal{I} \subset \mathcal{M}, \quad xz, zx \in \mathcal{I} \subset \mathcal{M};$$

moreover,

$$\mathbb{I} \in \bigcap_{\mathcal{I} \in C} (\mathcal{A} \setminus \mathcal{I}) = \mathcal{A} \setminus \bigcup_{\mathcal{I} \in C} \mathcal{I} = \mathcal{A} \setminus \mathcal{M},$$

so that $\mathcal{M} \neq \mathcal{A}$. We have proven that $\mathcal{M}$ is an ideal. The maximality of the chain C implies that $\mathcal{M}$ is maximal $\square$

Quotient algebra. Let $\mathcal{A}$ be an algebra with an ideal $\mathcal{J}$. For $x \in \mathcal{A}$, let us denote

$$[x] := x + \mathcal{J} = \{x + j \mid j \in \mathcal{J}\}.$$

Then the set $\mathcal{A}/\mathcal{J} := \{[x] \mid x \in \mathcal{A}\}$ can be endowed with a natural algebra structure: Let us define

$$\lambda[x] := [\lambda x], \quad [x] + [y] := [x + y], \quad [x][y] := [xy], \quad \mathbb{I}_{\mathcal{A}/\mathcal{J}} := [\mathbb{I}_{\mathcal{A}}];$$

first of all, these operations are well-defined, since if $\lambda \in \mathbb{C}$ and $j, j_1, j_2 \in \mathcal{J}$ then

$$\begin{aligned}
\lambda(x + j) &= \lambda x + \lambda j \in [\lambda x], \\
(x + j_1) + (y + j_2) &= (x + y) + (j_1 + j_2) \in [x + y], \\
(x + j_1)(y + j_2) &= xy + j_1 y + x j_2 + j_1 j_2 \in [xy].
\end{aligned}$$

Secondly, $[\mathbb{I}_{\mathcal{A}}] = \mathbb{I}_{\mathcal{A}} + \mathcal{J} \neq \mathcal{J} = [0]$, because $\mathbb{I}_{\mathcal{A}} \notin \mathcal{J}$. Moreover,

$$\begin{aligned}
(x + j_1)(\mathbb{I}_{\mathcal{A}} + j_2) &= x + j_1 + x j_2 + j_1 j_2 \in [x], \\
(\mathbb{I}_{\mathcal{A}} + j_2)(x + j_1) &= x + j_1 + j_2 x + j_2 j_1 \in [x].
\end{aligned}$$

Now the reader may verify that $\mathcal{A}/\mathcal{J}$ is really an algebra; it is called the *quotient algebra of* $\mathcal{A}$ *modulo* $\mathcal{J}$.

Remarks: Notice that $\mathcal{A}/\mathcal{J}$ is commutative if $\mathcal{A}$ is commutative. Also notice that $[0] = \mathcal{J}$ is the zero element in the quotient algebra.

Homomorphisms. Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. A mapping $\phi : \mathcal{A} \to \mathcal{B}$ is called a *homomorphism* if it is a linear mapping satisfying

$$\phi(xy) = \phi(x)\phi(y)$$

for every $x, y \in \mathcal{A}$ (multiplicativity) and

$$\phi(\mathbb{I}_{\mathcal{A}}) = \mathbb{I}_{\mathcal{B}}.$$

The set of all homomorphisms $\mathcal{A} \to \mathcal{B}$ is denoted by

$$\mathrm{Hom}(\mathcal{A}, \mathcal{B}).$$

A bijective homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is called an *isomorphism*, denoted by $\phi : \mathcal{A} \cong \mathcal{B}$.

Examples of homomorphisms.

1. The only homomorphism $\mathbb{C} \to \mathbb{C}$ is the identity mapping, $\mathrm{Hom}(\mathbb{C}, \mathbb{C}) = \{x \mapsto x\}$.

2. Let $x \in X$. Let us define the evaluation mapping $\phi_x : \mathcal{F}(X) \to \mathbb{C}$ by $f \mapsto f(x)$. Then $\phi_x \in \mathrm{Hom}(\mathcal{F}(X), \mathbb{C})$.

3. Let $\mathcal{J}$ be an ideal of an algebra $\mathcal{A}$, and denote $[x] = x + \mathcal{J}$. Then $(x \mapsto [x]) \in \mathrm{Hom}(\mathcal{A}, \mathcal{A}/\mathcal{J})$.

Exercise*. Let $\phi \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$. If $x \in \mathcal{A}$ is invertible then $\phi(x) \in \mathcal{B}$ is invertible. For any $x \in \mathcal{A}$, $\sigma_{\mathcal{B}}(\phi(x)) \subset \sigma_{\mathcal{A}}(x)$.

Exercise. Let $\mathcal{A}$ be the set of matrices

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \qquad (\alpha, \beta \in \mathbb{C}).$$

Show that $\mathcal{A}$ is a commutative algebra. Classify (up to an isomorphism) all the two-dimensional algebras. (Hint: Prove that in a two-dimensional algebra either $\exists x \neq 0 : \ x^2 = 0$ or $\exists x \notin \{\mathbb{I}, -\mathbb{I}\} : \ x^2 = \mathbb{I}$.)

Proposition. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras, and $\phi \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$. Then $\phi(\mathcal{A}) \subset \mathcal{B}$ is a subalgebra, $\mathrm{Ker}(\phi) := \{x \in \mathcal{A} \mid \phi(x) = 0\}$ is an ideal of $\mathcal{A}$, and $\mathcal{A}/\mathrm{Ker}(\phi) \cong \phi(\mathcal{A})$.*

Exercise*. Prove the previous Proposition.