Mat-1.152
Special Course in Functional Analysis: (Non-)Commutative Geometry

What is this good for? You may learn something about functional analytic framework of topology and measure theory. And you will get an access to more advanced literature on non-commutative geometry, a quite recent topic in mathematics and mathematical physics.

What the reader is assumed to know? The prerequisite for this course is some elementary understanding of Banach spaces. Of course, it helps if the reader already knows some topology and measure theory, but we shall explicitly introduce every major mathematical tool we need. We have carefully tried to keep the presentation as simple as possible. Well, the introducing section may contain many unfamiliar concepts, but do not worry: everything will be made precise.

References

[1] B. Aupetit: *A Primer on Spectral Theory.* Springer-Verlag 1991.

[2] A. Connes: *Noncommutative Geometry.* Academic Press 1994.

[3] R. G. Douglas: *Banach Algebra Techniques in Operator Theory.* Academic Press 1972.

[4] A. Friedmann: *Foundations of Modern Analysis.* Dover 1982.

[5] R. F. Gariepy, W. P. Ziemer: *Modern Real Analysis.* PWS Publishing Company 1995.

[6] J. M. Gracia-Bondía, J. C. Várilly, H. Figueroa: *Elements of Noncommutative Geometry.* Birkhäuser 2001.

[7] G. J. Murphy: *C^*-algebras and Operator Theory.* Academic Press 1990.

[8] H. L. Royden: *Real Analysis.* The Macmillan Company 1963.

[9] W. Rudin: *Functional Analysis.* Tata McGraw-Hill 1992.

[10] W. Rudin: *Real and Complex Analysis.* McGraw-Hill 1986.

[11] L. A. Steen and J. A. Seebach, Jr.: *Counterexamples in Topology.* Dover 1995.

[12] P. Suppes: *Axiomatic Set Theory.* Dover 1972.

[13] J. Väisälä: *Topologia II.* Limes 1987.

[14] N. Weaver: *Lipschitz Algebras.* World Scientific 1999.

[15] N. E. Wegge-Olsen: *K-Theory and C^*-algebras.* Oxford University Press 1993.

More information on the lecture topics:
Set theory and Axiom of Choice: [12, 10, 13].
Topology: [13, 11, 8, 5].
Measure theory: [4, 5, 8, 10].
Basic functional analysis: [10, 9].
Banach algebras and C*-algebras: [7, 1, 9, 3, 6] and practically any book on advanced functional analysis ($C^*=B^*$ in [9] :)
Lipschitz algebras: [14].

For those mastering these lecture notes:
Non-commutative geometry: [2, 6, 15].

1 Informal introduction

These lecture notes present a fundamental relationship between topology, measure theory and algebra. Briefly, if we want to study properties of a space X, we may alternatively examine some algebra of functions on X. With suitable topological restrictions, there will be a bijective correspondence between spaces and algebras (equivalence of categories, if you insist). Topology and measure theory of X can then be stated in the terms of a topological function algebra. And it will turn out that the tools that are developed for the study of function algebras work as well for non-commutative algebras.

Let us begin with a trivial example. Let X be a finite set. Let $\mathcal{A} = \mathcal{F}(X)$ be the set of the complex-valued functions $f : X \to \mathbb{C}$. Then $\mathcal{A}$ is naturally a $\mathbb{C}$-vector space:

$$(f+g)(x) := f(x) + g(x), \qquad (\lambda f)(x) := \lambda\ f(x)$$

for every $f, g \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Moreover, $\mathcal{A}$ is an algebra when endowed with the product

$$(fg)(x) := f(x)\ g(x)$$

and with the unit element $\mathbb{I} = \mathbb{I}_{\mathcal{A}}$, which is the constant function $\mathbb{I}(x) \equiv 1$. Let $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ denote the set of algebra homomorphisms $\mathcal{A} \to \mathbb{C}$. When $x \in X$, the evaluation mapping

$$(f \mapsto f(x)) : \mathcal{A} \to \mathbb{C}$$

is a homomorphism. Hence we may think that X is a subset of $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$. Actually, it turns out that the evaluation mappings are the only homomorphisms $\mathcal{A} \to \mathbb{C}$. Therefore we may even claim

$$X = \mathrm{Hom}(\mathcal{A}, \mathbb{C}).$$

It can be proven that every isomorphism $\mathcal{A} \to \mathcal{A}$ arises from a bijection $X \to X$. And if $\emptyset \neq I \subset X$, it is easy to see that

$$\mathcal{I} := \{f \in \mathcal{A} \mid \forall x \in I : f(x) = 0\}$$

is an ideal of the algebra $\mathcal{A}$, and that there are no other ideals; thus the non-empty subsets of X are in bijective correspondence with the ideals of $\mathcal{A}$. Any $\lambda : X \to \mathbb{C}$ defines a linear functional $\Lambda : \mathcal{A} \to \mathbb{C}$ by

$$\Lambda f := \sum_{x \in X} f(x)\ \lambda(x),$$

which can be thought as an integral with respect to the λ-weighted counting measure on X; conversely, any linear functional on $\mathcal{A}$ arises this way.

From analysis point of view, the discrete topology is the most reasonable topology for a finite set X, and the counting measure is the natural choice for measure theory. We should not endow an infinite set with the discrete topology nor with the counting measure. Instead, non-trivial topology and measure theory will be necessary. The framework is the theory of commutative C*-algebras ("C-star-algebra"), an extremely beautiful branch of functional analysis. In essence, this theory boils down to the following:

Theorem. *Compact Hausdorff spaces X and Y are homeomorphic if and only if the function algebras $C(X)$ and $C(Y)$ of complex-valued continuous functions are isomorphic.*

Thus the topological and measure theoretic information of some topological space X is equivalent to the topologic-algebraic information of $C(X)$. The same phenomenon occurs also for differentiability properties. We may study directly the geometry of a space, but as well we may study algebras of functions on it! This is called **"commutative geometry"**, as function algebras are commutative. Now this remark almost forces us to generalize: We may study certain non-commutative algebras using similar tools as in the commutative case. Hence the name **"non-commutative geometry"**.

The reader may wonder why these themes should be relevant. We have already expressed the nice connections between different branches of mathematics. Let us go back in the history: In 1925, Werner Heisenberg (1901-1976) and Erwin Schrödinger (1887-1961) initiated the quantum mechanics. Heisenberg applied matrix algebras, while Schrödinger practically studied Fourier analysis, but their theories were essentially equivalent. However, a precise mathematical foundation for quantum physics was lacking. This was the main reason for János von Neumann (1903-1957) to develop the Hilbert spaces and the spectral theory of normal operators in 1929-1930. In this context, a quantum mechanical system is presented as a partial differential equation (*Schrödinger equation*) on a Hilbert space $\mathcal{H}$, where unit vectors $\phi \in \mathcal{H}$ ($\|\phi\| = 1$) are *states* of the system. The measurable quantities, or *observables*, of the system are the self-adjoint linear operators ($A^* = A$) on $\mathcal{H}$. Also the unbounded operators are interesting, for instance the location and momentum operators on $L^2(\mathbb{R}^n)$. When we measure a quantity, the result is not the full information about the observable but merely a value from the spectrum of the operator (e.g. try to locate a particle in the space). The interesting thing is then to find a spectral decomposition of an observable, analogous to the diagonalization of a Hermitian matrix.

Well, this is not a physical Theory of Everything. Anyhow, $\mathcal{L}(\mathcal{H})$ is the natural first stage in developing the operator algebras. In 1936, the next ingenious step was the theory of von Neumann algebras, capturing some measure theoretic properties of classical L^∞-type spaces. These algebras were a special case of C*-algebras, whose theory emerged in the early of 1940s mainly by Israil Gelfand (1913-).

Practical definition. *A* C*-algebra $\mathcal{A}$ *is a norm closed involutive subalgebra of* $\mathcal{L}(\mathcal{H})$ *for some Hilbert space* $\mathcal{H}$.

Equivalent abstract definition. *A* C*-*algebra* $\mathcal{A}$ *is a Banach space and a* $\mathbb{C}$-*algebra with involution* $x \mapsto x^*$ *such that*

$$\|xy\| \leq \|x\| \, \|y\|, \quad \|\mathbb{I}_{\mathcal{A}}\| = 1, \quad \|x^*x\| = \|x\|^2$$

for every $x, y \in \mathcal{A}$.

Gelfand's idea was to look at a "mirror reflection" of a commutative algebra. Actually, this approach can be dated back at least to Hilbert's *Nullstellensatz* in algebraic geometry, in 1893. Let $\mathcal{A}$ be a commutative C*-algebra. Let $X = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$ be the set of algebra homomorphisms $\mathcal{A} \to \mathbb{C}$. The *Gelfand transform* of an element $f \in \mathcal{A}$ is the function $\widehat{f} : X \to \mathbb{C}$ defined by

$$\widehat{f}(x) := x(f),$$

where $x \in X = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$. This seems astonishingly simple, but is fundamental. Now Gelfand proved that X is a compact Hausdorff space in the natural topology inherited from the weak*-topology of the dual space $\mathcal{A}' = \mathcal{L}(\mathcal{A}, \mathbb{C})$. Moreover:

Gelfand–Naimark Theorem (1943). *Any commutative C*-algebra is isometrically isomorphic to the algebra* $C(X)$ *for some compact Hausdorff space* X.

Gelfand–Naimark Theorem is the starting point of the non-commutative geometry, which was initiated by Alain Connes in the 1980s. By now, this huge subject contains such topics as Hopf algebras and quantum groups, K-theory for operator algebras, non-commutative integrodifferential calculus, non-commutative manifolds, and so on. This machinery has been applied e.g. in particle physics, quantum field theory and string theory.

However, within the limited time we have, we only present some of the early fundamental results of topology and operator algebras. This hopefully provides a solid background for the reader to investigate non-commutative geometry further.

To distill some of the essential results that will be obtained in these lecture notes, we present a "dictionary" relating topology, measure theory and algebra; here X is a compact Hausdorff space:

Topology / Measure theory	$\leftrightarrow$	**Algebra**
$C(X)$	$\leftrightarrow$	commutative C*-algebra $\mathcal{A}$
homeomorphism $X \to X$	$\leftrightarrow$	isomorphism $\mathcal{A} \to \mathcal{A}$
point $\in X$	$\leftrightarrow$	maximal ideal $\in \mathcal{A}$ or homomorphism $\mathcal{A} \to \mathbb{C}$
non-empty closed subset $\subset X$	$\leftrightarrow$	closed ideal $\subset \mathcal{A}$ or quotient algebra $\mathcal{A}$/ideal
range of $f \in C(X)$	$\leftrightarrow$	spectrum of an element $\in \mathcal{A}$
X metrizable	$\leftrightarrow$	$\mathcal{A}$ separable
X disconnected	$\leftrightarrow$	$\exists f \in \mathcal{A}: \ f^2 = f, 0 \neq f \neq 1$
complex measure on X	$\leftrightarrow$	bounded linear functional $\mathcal{A} \to \mathbb{C}$
positive measure on X	$\leftrightarrow$	positive linear functional $\mathcal{A} \to \mathbb{C}$
$\vdots$	$\leftrightarrow$	$\vdots$

Appendix on set theoretical notation

When X is a set, $\mathcal{P}(X)$ denotes the family of all subsets of X (the *power set*, sometimes denoted by 2^X). The cardinality of X is denoted by $|X|$. If J is a set and $S_j \subset X$ for every $j \in J$, we write

$$\bigcup \{S_j \mid j \in J\} = \bigcup_{j \in J} S_j, \quad \bigcap \{S_j \mid j \in J\} = \bigcap_{j \in J} S_j.$$

If $f : X \to Y$, $U \subset X$, and $V \subset Y$, we define

$$f(U) := \{f(x) \mid x \in U\} \quad \text{(image)},$$

$$f^{-1}(V) := \{x \in X \mid f(x) \in V\} \quad \text{(preimage)}.$$

Appendix on Axiom of Choice

It may be surprising, but the Zermelo-Fraenkel axiom system does not imply the following statement (nor its negation):

Axiom of Choice for Cartesian Products: *The Cartesian product of non-empty sets is non-empty.*

Nowadays there are hundreds of equivalent formulations for the Axiom of Choice. Next we present other famous variants: the classical Axiom of Choice, the Law of Trichotomy, the Well-Ordering Axiom, the Hausdorff Maximal Principle and Zorn's Lemma. Their equivalence is shown in [12].

Axiom of Choice: *For every non-empty set J there is a function $f : \mathcal{P}(J) \to J$ such that $f(I) \in I$ when $I \neq \emptyset$.*

Let A, B be sets. We write $A \sim B$ if there exists a bijection $f : A \to B$, and $A \leq B$ if there is a set $C \subset B$ such that $A \sim C$. Notion $A < B$ means $A \leq B$ such that not $A \sim B$.

Law of Trichotomy: *Let A, B be sets. Then $A < B$, $A \sim B$ or $B < A$.*

A set X is *partially ordered* with an *order relation* $R \subset X \times X$ if R is reflexive ($(x,x) \in R$), antisymmetric ($(x,y),(y,x) \in R \Rightarrow x = y$) and transitive ($(x,y),(y,z) \in R \Rightarrow (x,z) \in R$). A subset $C \subset X$ is a *chain* if $(x,y) \in R$ or $(y,x) \in R$ for every $x, y \in C$. An element $x \in X$ is *maximal* if $(x,y) \in R$ implies $y = x$.

Well-Ordering Axiom: *Every set is a chain for some order relation.*

Hausdorff Maximal Principle: *Any chain is contained in a maximal chain.*

Zorn's Lemma: *A non-empty partially ordered set where every chain has an upper bound has a maximal element.*

2 Algebras

Algebra. A vector space $\mathcal{A}$ over the field $\mathbb{C}$ is an *algebra* if there exists an element $\mathbb{I}_{\mathcal{A}} \in \mathcal{A} \setminus \{0\}$ and a mapping $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(x, y) \mapsto xy$, satisfying

$$\begin{aligned} &x(yz) = (xy)z, \\ &x(y+z) = xy + xz, \quad (x+y)z = xz + yz, \\ &\lambda(xy) = (\lambda x)y = x(\lambda y), \\ &\mathbb{I}_{\mathcal{A}} x = x = x \mathbb{I}_{\mathcal{A}} \end{aligned}$$

for every $x, y, z \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. We briefly write $xyz := x(yz)$. The element $\mathbb{I} := \mathbb{I}_{\mathcal{A}}$ is called the *unit of* $\mathcal{A}$, and an element $x \in \mathcal{A}$ is called *invertible* (with the unique *inverse* x^{-1}) if there exists $x^{-1} \in \mathcal{A}$ such that

$$x^{-1}x = \mathbb{I} = xx^{-1}.$$

If $xy = yx$ for every $x, y \in \mathcal{A}$ then $\mathcal{A}$ is called *commutative.*

Warnings: In some books the algebra axioms allow $\mathbb{I}_{\mathcal{A}}$ to be 0, but then the resulting algebra is simply $\{0\}$; we have omitted such a triviality. In some books the existence of a unit is omitted from the algebra axioms; what we have called an algebra is there called a *unital algebra.*

Examples of algebras.

1. $\mathbb{C}$ is the most important algebra. The operations are the usual ones for complex numbers, and the unit element is $\mathbb{I}_{\mathbb{C}} = 1 \in \mathbb{C}$. Clearly $\mathbb{C}$ is a commutative algebra.

2. The algebra $\mathcal{F}(X) := \{f \mid f : X \to \mathbb{C}\}$ of complex valued functions on a (finite or infinite) set X is endowed with the same algebra structure as in the example in "Informal introduction" section (pointwise operations). Function algebras are commutative, because $\mathbb{C}$ is commutative.

3. The algebra $L(V) := \{A : V \to V \mid A \text{ is linear}\}$ of linear operators on a vector space $V \neq \{0\}$ over $\mathbb{C}$ is endowed with the usual vector space structure and with the multiplication $(A, B) \mapsto AB$ (composition of operators); the unit element is $\mathbb{I}_{L(V)} = (v \mapsto v) : V \to V$, the identity operator on V. This algebra is non-commutative if V is at least two-dimensional.

Exercise. Let $\mathcal{A}$ be an algebra and $x, y \in \mathcal{A}$. Prove the following claims:
(a) If x, xy are invertible then y is invertible.
(b) If xy, yx are invertible then x, y are invertible.

Exercise. Give an example of an algebra $\mathcal{A}$ and elements $x, y \in \mathcal{A}$ such that $xy = \mathbb{I}_{\mathcal{A}} \neq yx$. Prove that then $(yx)^2 = yx \neq 0$. (Hint: Such an algebra is necessarily infinite-dimensional).

Spectrum. Let $\mathcal{A}$ be an algebra. The *spectrum* $\sigma(x)$ of an element $x \in \mathcal{A}$ is the set

$$\sigma_{\mathcal{A}}(x) = \sigma(x) = \{\lambda \in \mathbb{C} : \ \lambda\mathbb{I} - x \text{ is not invertible}\}.$$

Examples of invertibility and spectra.

1. An element $\lambda \in \mathbb{C}$ is invertible if and only if $\lambda \neq 0$; the inverse of an invertible λ is the usual $\lambda^{-1} = 1/\lambda$. Generally, $\sigma_{\mathbb{C}}(\lambda) = \{\lambda\}$.

2. An element $f \in \mathcal{F}(X)$ is invertible if and only if $f(x) \neq 0$ for every $x \in X$. The inverse of an invertible f is g with $g(x) = f(x)^{-1}$. Generally, $\sigma_{\mathcal{F}(X)}(f) = f(X) := \{f(x) \mid x \in X\}$.

3. An element $A \in L(V)$ is invertible if and only if it is a bijection (if and only if $0 \notin \sigma_{L(V)}(A)$).

Exercise. Let $\mathcal{A}$ be an algebra and $x, y \in \mathcal{A}$. Prove the following claims:
(a) $\mathbb{I} - yx$ is invertible if and only if $\mathbb{I} - xy$ is invertible.
(b) $\sigma(yx) \subset \sigma(xy) \cup \{0\}$.
(c) If x is invertible then $\sigma(xy) = \sigma(yx)$.

Ideals. Let $\mathcal{A}$ be an algebra. An *ideal* $\mathcal{J} \subset \mathcal{A}$ is a vector subspace $\mathcal{J} \neq \mathcal{A}$ satisfying

$$\forall x \in \mathcal{A} \ \forall y \in \mathcal{J} : \ xy, yx \in \mathcal{J},$$

i.e. $x\mathcal{J}, \mathcal{J}x \subset \mathcal{J}$ for every $x \in \mathcal{A}$. A *maximal ideal* is an ideal not contained in any other ideal.

Warning. In some books our ideals are called *proper ideals*, and there *ideal* is either a proper ideal or the whole algebra.

Remark. Let $\mathcal{J} \subset \mathcal{A}$ be an ideal. Because $x\mathbb{I} = x$ for every $x \in \mathcal{A}$, we notice that $\mathbb{I} \notin \mathcal{J}$. Therefore an invertible element $x \in \mathcal{A}$ cannot belong to an ideal (since $x^{-1}x = \mathbb{I} \notin \mathcal{J}$).

Examples of ideals. Intuitively, an ideal of an algebra is a subspace resembling a multiplicative zero; consider equations $x0 = 0 = 0x$.

1. Let $\mathcal{A}$ be an algebra. Then $\{0\} \subset \mathcal{A}$ is an ideal.

2. The only ideal of $\mathbb{C}$ is $\{0\} \subset \mathbb{C}$.

3. Let X be a set, and $\emptyset \neq S \subset X$. Now

$$\mathcal{I}(S) := \{f \in \mathcal{F}(X) \mid \forall x \in S: \ f(x) = 0\}$$

   is an ideal of the function algebra $\mathcal{F}(X)$. If $x \in X$ then $\mathcal{I}(\{x\})$ is a maximal ideal of $\mathcal{F}(X)$, because it is of co-dimension 1 in $\mathcal{F}(X)$. Notice that $\mathcal{I}(S) \subset \mathcal{I}(\{x\})$ for every $x \in S$; an ideal may be contained in many different maximal ideals (cf. Krull's Theorem in the sequel).

4. Let X be an infinite-dimensional Banach space. The set

$$\mathcal{LC}(X) := \{A \in \mathcal{L}(X) \mid A \text{ is compact}\}$$

   of compact linear operators $X \to X$ is an ideal of the algebra $\mathcal{L}(X)$ of bounded linear operators $X \to X$.

Theorem (W. Krull). *An ideal is contained in a maximal ideal.*

Proof. Let $\mathcal{J}$ be an ideal of an algebra $\mathcal{A}$. Let P be the set of those ideals of $\mathcal{A}$ that contain $\mathcal{J}$. The inclusion relation is the natural partial order on P; the **Hausdorff Maximal Principle** says that there is a maximal chain $C \subset P$. Let $\mathcal{M} := \bigcup C$. Clearly $\mathcal{J} \subset \mathcal{M}$. Let $\lambda \in \mathbb{C}$, $x, y \in \mathcal{M}$ and $z \in \mathcal{A}$. Then there exists $\mathcal{I} \in C$ such that $x, y \in \mathcal{I}$, so that

$$\lambda x \in \mathcal{I} \subset \mathcal{M}, \quad x + y \in \mathcal{I} \subset \mathcal{M}, \quad xz, zx \in \mathcal{I} \subset \mathcal{M};$$

moreover,

$$\mathbb{I} \in \bigcap_{\mathcal{I} \in C} (\mathcal{A} \setminus \mathcal{I}) = \mathcal{A} \setminus \bigcup_{\mathcal{I} \in C} \mathcal{I} = \mathcal{A} \setminus \mathcal{M},$$

so that $\mathcal{M} \neq \mathcal{A}$. We have proven that $\mathcal{M}$ is an ideal. The maximality of the chain C implies that $\mathcal{M}$ is maximal $\square$

Quotient algebra. Let $\mathcal{A}$ be an algebra with an ideal $\mathcal{J}$. For $x \in \mathcal{A}$, let us denote

$$[x] := x + \mathcal{J} = \{x + j \mid j \in \mathcal{J}\}.$$

Then the set $\mathcal{A}/\mathcal{J} := \{[x] \mid x \in \mathcal{A}\}$ can be endowed with a natural algebra structure: Let us define

$$\lambda[x] := [\lambda x], \quad [x] + [y] := [x + y], \quad [x][y] := [xy], \quad \mathbb{I}_{\mathcal{A}/\mathcal{J}} := [\mathbb{I}_{\mathcal{A}}];$$

first of all, these operations are well-defined, since if $\lambda \in \mathbb{C}$ and $j, j_1, j_2 \in \mathcal{J}$ then

$$\begin{aligned} \lambda(x + j) &= \lambda x + \lambda j \in [\lambda x], \\ (x + j_1) + (y + j_2) &= (x + y) + (j_1 + j_2) \in [x + y], \\ (x + j_1)(y + j_2) &= xy + j_1 y + x j_2 + j_1 j_2 \in [xy]. \end{aligned}$$

Secondly, $[\mathbb{I}_{\mathcal{A}}] = \mathbb{I}_{\mathcal{A}} + \mathcal{J} \neq \mathcal{J} = [0]$, because $\mathbb{I}_{\mathcal{A}} \notin \mathcal{J}$. Moreover,

$$\begin{aligned} (x + j_1)(\mathbb{I}_{\mathcal{A}} + j_2) &= x + j_1 + x j_2 + j_1 j_2 \in [x], \\ (\mathbb{I}_{\mathcal{A}} + j_2)(x + j_1) &= x + j_1 + j_2 x + j_2 j_1 \in [x]. \end{aligned}$$

Now the reader may verify that $\mathcal{A}/\mathcal{J}$ is really an algebra; it is called the *quotient algebra of $\mathcal{A}$ modulo $\mathcal{J}$.*

Remarks: Notice that $\mathcal{A}/\mathcal{J}$ is commutative if $\mathcal{A}$ is commutative. Also notice that $[0] = \mathcal{J}$ is the zero element in the quotient algebra.

Homomorphisms. Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. A mapping $\phi : \mathcal{A} \to \mathcal{B}$ is called a *homomorphism* if it is a linear mapping satisfying

$$\phi(xy) = \phi(x)\phi(y)$$

for every $x, y \in \mathcal{A}$ (multiplicativity) and

$$\phi(\mathbb{I}_{\mathcal{A}}) = \mathbb{I}_{\mathcal{B}}.$$

The set of all homomorphisms $\mathcal{A} \to \mathcal{B}$ is denoted by

$$\mathrm{Hom}(\mathcal{A}, \mathcal{B}).$$

A bijective homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is called an *isomorphism*, denoted by $\phi : \mathcal{A} \cong \mathcal{B}$.

Examples of homomorphisms.

1. The only homomorphism $\mathbb{C} \to \mathbb{C}$ is the identity mapping, $\mathrm{Hom}(\mathbb{C}, \mathbb{C}) = \{x \mapsto x\}$.

2. Let $x \in X$. Let us define the evaluation mapping $\phi_x : \mathcal{F}(X) \to \mathbb{C}$ by $f \mapsto f(x)$. Then $\phi_x \in \mathrm{Hom}(\mathcal{F}(X), \mathbb{C})$.

3. Let $\mathcal{J}$ be an ideal of an algebra $\mathcal{A}$, and denote $[x] = x + \mathcal{J}$. Then $(x \mapsto [x]) \in \mathrm{Hom}(\mathcal{A}, \mathcal{A}/\mathcal{J})$.

Exercise*. Let $\phi \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$. If $x \in \mathcal{A}$ is invertible then $\phi(x) \in \mathcal{B}$ is invertible. For any $x \in \mathcal{A}$, $\sigma_{\mathcal{B}}(\phi(x)) \subset \sigma_{\mathcal{A}}(x)$.

Exercise. Let $\mathcal{A}$ be the set of matrices

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \qquad (\alpha, \beta \in \mathbb{C}).$$

Show that $\mathcal{A}$ is a commutative algebra. Classify (up to an isomorphism) all the two-dimensional algebras. (Hint: Prove that in a two-dimensional algebra either $\exists x \neq 0 : \ x^2 = 0$ or $\exists x \notin \{\mathbb{I}, -\mathbb{I}\} : \ x^2 = \mathbb{I}$.)

Proposition. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras, and $\phi \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$. Then $\phi(\mathcal{A}) \subset \mathcal{B}$ is a subalgebra,* $\mathrm{Ker}(\phi) := \{x \in \mathcal{A} \mid \phi(x) = 0\}$ *is an ideal of $\mathcal{A}$, and* $\mathcal{A}/\mathrm{Ker}(\phi) \cong \phi(\mathcal{A})$.

Exercise*. Prove the previous Proposition.

3 Topology (and metric), basics

The reader should know metric spaces; topological spaces are their generalization, which we soon introduce. Feel free to draw some clarifying schematic pictures on the margins!

Metric space. A function $d : X \times X \to [0, \infty[$ is called a *metric* on the set X if for every $x, y, z \in X$ we have

- $d(x,y) = 0 \Leftrightarrow x = y$;
- $d(x,y) = d(y,x)$;
- $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

Then (X, d) (or simply X when d is evident) is called a *metric space*. Sometimes a metric is called a *distance function*.

Topological space. A family of sets $\tau \subset \mathcal{P}(X)$ is called a *topology* on the set X if

1. $\emptyset, X \in \tau$;
2. $\mathcal{U} \subset \tau \Rightarrow \bigcup \mathcal{U} \in \tau$;
3. $U, V \in \tau \Rightarrow U \cap V \in \tau$.

Then (X, τ) (or simply X when τ is evident) is called a *topological space*. The sets $U \in \tau$ are called *open sets*, and their complements $X \setminus U$ are *closed sets*.

Thus in a topological space, the empty set and the whole space are always open, **any** union of open sets is open, and an intersection of **finitely** many open sets is open. Equivalently, the whole space and the empty set are always closed, **any** intersection of closed sets is closed, and a union of **finitely** many closed sets is closed.

Metric topology. Let (X, d) be a metric space. We say that the *open ball of radius $r > 0$ centered at $x \in X$* is

$$B_d(x,r) := \{y \in X \mid d(x,y) < r\}.$$

The *metric topology* τ_d of (X, d) is given by

$$U \in \tau_d \overset{\text{definition}}{\Leftrightarrow} \forall x \in U \ \exists r > 0 : \ B_d(x,r) \subset U.$$

A topological space (X, τ) is called *metrizable* if there is a metric d on X such that $\tau = \tau_d$.

Non-metrizable spaces. There are plenty of non-metrizable topological spaces, the easiest example being X with more than one point and with $\tau = \{\emptyset, X\}$. If X is an infinite-dimensional Banach space then the weak*-topology of $X' := \mathcal{L}(X, \mathbb{C})$ is not metrizable. The distribution spaces $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ are non-metrizable topological spaces. We shall later prove that for the compact Hausdorff spaces metrizability is equivalent to the existence of a countable base.

Base. Let (X, τ) be a topological space. A family $\mathcal{B} \subset \tau$ of open sets is called a *base* (or *basis*) for the topology τ if any open set is a union of some members of $\mathcal{B}$, i.e.

$$\forall U \in \tau \ \exists \mathcal{B}' \subset \ \mathcal{B} : \ U = \bigcup \mathcal{B}'.$$

Examples. Trivially a topology τ is a base for itself ($\forall U \in \tau : U = \bigcup \{U\}$). If (X, d) is a metric space then

$$\mathcal{B} := \{B_d(x, r) \mid x \in X, \ r > 0\}$$

constitutes a base for τ_d.

Neighborhoods. Let (X, τ) be a topological space. A *neighborhood of* $x \in X$ is any open set $U \subset X$ containing x. The family of neighborhoods of $x \in X$ is denoted by

$$\mathcal{V}_\tau(x) := \{U \in \tau \mid x \in U\}$$

(or simply $\mathcal{V}(x)$, when τ is evident).

The natural mappings (or the morphisms) between topological spaces are **continuous mappings**.

Continuity at a point. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A mapping $f : X \to Y$ is *continuous at* $x \in X$ if

$$\forall V \in \mathcal{V}_{\tau_Y}(f(x)) \ \exists U \in \mathcal{V}_{\tau_X}(x) : \ f(U) \subset V.$$

Exercise. Let (X, d_X) and (Y, d_Y) be metric spaces. A mapping $f : X \to Y$ is continuous at $x \in X$ if and only if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in X : \ d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$$

if and only if

$$d_X(x_n, x) \to_{n\to\infty} 0 \Rightarrow d_Y(f(x_n), f(x)) \to_{n\to\infty} 0$$

for every sequence $(x_n)_{n=1}^{\infty} \subset X$ (that is, $x_n \to x \Rightarrow f(x_n) \to f(x)$).

Continuity. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A mapping $f : X \to Y$ is *continuous*, denoted by $f \in C(X, Y)$, if

$$\forall V \in \tau_Y : f^{-1}(V) \in \tau_X,$$

where $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$; i.e. f is continuous if preimages of open sets are open (equivalently, preimages of closed sets are closed). In the sequel, we briefly write

$$C(X) := C(X, \mathbb{C}),$$

where $\mathbb{C}$ has the metric topology with the usual metric $(\lambda, \mu) \mapsto |\lambda - \mu|$.

Proposition. *Let (X, τ_X) and (Y, τ_Y) be topological spaces. A mapping $f : X \to Y$ is continuous at every $x \in X$ if and only if it is continuous.*

Proof. Suppose $f : X \to Y$ is continuous, $x \in X$, and $V \in \mathcal{V}_{\tau_Y}(f(x))$. Then $U := f^{-1}(V)$ is open, $x \in U$, and $f(U) = V$, implying the continuity at $x \in X$.

Conversely, suppose $f : X \to Y$ is continuous at every $x \in X$, and let $V \subset Y$ be open. Choose $U_x \in \mathcal{V}_{\tau_X}(x)$ such that $f(U_x) \subset V$ for every $x \in f^{-1}(V)$. Then

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$$

is open in X □

Exercise. Let X be a topological space. Show that $C(X)$ is an algebra.

Exercise. Prove that if $f : X \to Y$ and $g : Y \to Z$ are continuous then $g \circ f : X \to Z$ is continuous.

Topological equivalence: homeomorphism. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A mapping $f : X \to Y$ is called a *homeomorphism* if it is a bijection, $f \in C(X,Y)$ and $f^{-1} \in C(Y,X)$. Then X and Y are called *homeomorphic* or *topologically equivalent*, denoted by $X \cong Y$ or $f : X \cong Y$; more specifically, $f : (X, \tau_X) \cong (Y, \tau_Y)$.

Note that from the topology point of view, homeomorphic spaces can be considered equal.

Examples. Of course $(x \mapsto x) : (X, \tau) \cong (X, \tau)$. The reader may check that $(x \mapsto x/(1+|x|)) : \mathbb{R} \cong]-1,1[$. Using algebraic topology, one can prove that $\mathbb{R}^m \cong \mathbb{R}^n$ if and only if $m = n$ (this is not trivial!).

Metric equivalence and isometries. Metrics d_1, d_2 on a set X are called *equivalent* if there exists $m < \infty$ such that

$$M^{-1}\ d_1(x,y) \leq d_2(x,y) \leq M\ d_1(x,y)$$

for every $x, y \in X$. An *isometry* between metric spaces (X, d_X) and (Y, d_Y) is a mapping $f : X \to Y$ satisfying $d_Y(f(x), f(y)) = d_X(x,y)$ for every $x, y \in X$; f is called an *isometric isomorphism* if it is a surjective isometry (hence a bijection with an isometric isomorphism as the inverse mapping).

Examples. Any isometric isomorphism is a homeomorphism. Clearly the unbounded $\mathbb{R}$ and the bounded $]-1,1[$ are not isometrically isomorphic. An orthogonal linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is an isometric isomorphism, when $\mathbb{R}^n$ is endowed with the Euclidean norm. The forward shift operator on $\ell^p(\mathbb{Z})$ is an isometric isomorphism, but the forward shift operator on $\ell^p(\mathbb{N})$ is only a non-surjective isometry.

Hausdorff space. A topological space (X, τ) is a *Hausdorff space* if any two distinct points have some disjoint neighborhoods, i.e.

$$\forall x, y \in X\ \exists U \in \mathcal{V}(x)\ \exists V \in \mathcal{V}(y) :\ x \neq y \Rightarrow U \cap V = \emptyset.$$

Examples.

1. If τ_1 and τ_2 are topologies of X, $\tau_1 \subset \tau_2$, and (X, τ_1) is a Hausdorff space then (X, τ_2) is a Hausdorff space.

2. $(X, \mathcal{P}(X))$ is a Hausdorff space.

3. If X has more than one point and $\tau = \{\emptyset, X\}$ then (X, τ) is not Hausdorff.

4. Clearly any metric space (X, d) is a Hausdorff space; if $x, y \in X$, $x \neq y$, then $B_d(x, r) \cap B_d(y, r) = \emptyset$, when $r \leq d(x, y)/2$.

5. The distribution spaces $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ are non-metrizable Hausdorff spaces.

Exercise*. Let X be a Hausdorff space and $x \in X$. Then $\{x\} \subset X$ is a closed set.

Finite product topology. Let X, Y be topological spaces with bases $\mathcal{B}_X, \mathcal{B}_Y$, respectively. Then a base for the *product topology* of $X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$ is

$$\{U \times V \mid U \in \mathcal{B}_X,\ V \in \mathcal{B}_Y\}.$$

Exercise. Let X, Y be metrizable. Prove that $X \times Y$ is metrizable, and that

$$(x_n, y_n) \overset{X \times Y}{\to} (x, y) \quad \Leftrightarrow \quad x_n \overset{X}{\to} x \text{ and } y_n \overset{Y}{\to} y.$$

Closure and boundary. Let (X, τ) be a topological space. Let $S \subset X$; its *closure* $\mathrm{cl}_\tau(S) = \overline{S}$ is the smallest closed set containing S. The set S is *dense* in X if $\overline{S} = X$. The *boundary* of S is $\partial_\tau S = \partial S := \overline{S} \cap \overline{X \setminus S}$.

Exercise. Let (X, τ) be a topological space. Let $S, S_1, S_2 \subset X$. Show that
(a) $\overline{\emptyset} = \emptyset$,
(b) $S \subset \overline{S}$,
(c) $\overline{\overline{S}} = \overline{S}$,
(d) $\overline{S_1 \cup S_2} = \overline{S_1} \cup \overline{S_2}$.

Exercise. Let X be a set, $S, S_1, S_2 \subset X$. Let $c : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfy Kuratowski's closure axioms (a-d):
(a) $c(\emptyset) = \emptyset$,
(b) $S \subset c(S)$,
(c) $c(c(S)) = c(S)$,
(d) $c(S_1 \cup S_2) = c(S_1) \cup c(S_2)$.
Show that $\tau := \{U \subset X \mid c(X \setminus U) = X \setminus U\}$ is a topology of X, and that $\mathrm{cl}_\tau(S) = c(S)$ for every $S \subset X$.

Exercise. Let (X, τ) be a topological space. Prove that
(a) $x \in \overline{S} \;\Leftrightarrow\; \forall U \in \mathcal{V}(x) : \; U \cap S \neq \emptyset$.
(b) $\overline{S} = S \cup \partial S$.

4 Topological algebras

Topological algebra. A topological space and an algebra $\mathcal{A}$ is called a *topological algebra* if

1. $\{0\} \subset \mathcal{A}$ is a closed subset, and
2. the algebraic operations are continuous, i.e. the mappings

$$\begin{aligned} ((\lambda, x) \mapsto \lambda x): &\quad \mathbb{C} \times \mathcal{A} \to \mathcal{A}, \\ ((x, y) \mapsto x + y): &\quad \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \\ ((x, y) \mapsto xy): &\quad \mathcal{A} \times \mathcal{A} \to \mathcal{A} \end{aligned}$$

   are continuous.

Remark 1. Similarly, a *topological vector space* is a topological space and a vector space, in which $\{0\}$ is a closed subset and the vector space operations $(\lambda, x) \mapsto \lambda x$ and $(x, y) \mapsto x + y$ are continuous. And the reader might now guess how to define for instance a *topological group*...

Remark 2. Some books omit the assumption that $\{0\}$ should be a closed set; then e.g. any algebra $\mathcal{A}$ with a topology $\tau = \{\emptyset, \mathcal{A}\}$ would become a topological algebra. However, such generalizations are seldom useful. And it will turn out soon, that actually our topological algebras are indeed Hausdorff spaces! $\{0\}$ being a closed set puts emphasis on closed ideals and continuous homomorphisms, as we shall see later in this section.

Examples of topological algebras.

1. The commutative algebra $\mathbb{C}$ endowed with its usual topology (given by the absolute value norm $x \mapsto |x|$) is a topological algebra.
2. If $(X, x \mapsto \|x\|)$ is a normed space, $X \neq \{0\}$, then $\mathcal{L}(X)$ is a topological algebra with the norm

$$A \mapsto \|A\| := \sup_{x \in X : \|x\| \leq 1} \|Ax\|.$$

   Notice that $\mathcal{L}(\mathbb{C}) \cong \mathbb{C}$, and $\mathcal{L}(X)$ is non-commutative if $\dim(X) \geq 2$.
3. Let X be a set. Then

$$\mathcal{F}_b(X) := \{f \in \mathcal{F}(X) \mid f \text{ is bounded}\}$$

is a commutative topological algebra with the supremum norm

$$f \mapsto \|f\| := \sup_{x \in X} |f(x)|.$$

Similarly, if X is a topological space then the algebra

$$C_b(X) := \{f \in C(X) \mid f \text{ is bounded}\}$$

of bounded continuous functions on X is a commutative topological algebra when endowed with the supremum norm.

4. If (X, d) is a metric space then the algebra

$$\mathrm{Lip}(X) := \{f : X \to \mathbb{C} \mid f \text{ is Lipschitz continuous and bounded}\}$$

is a commutative topological algebra with the norm

$$f \mapsto \|f\| := \max \left\{ \sup_{x \in X} |f(x)|,\ \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} \right\}.$$

5. $\mathcal{E}(\mathbb{R}) := C^\infty(\mathbb{R})$ is a commutative topological algebra with the metric

$$(f, g) \mapsto \sum_{m=1}^{\infty} 2^{-m} \frac{p_m(f-g)}{1 + p_m(f-g)}, \quad \text{where } p_m(f) := \max_{|x| \leq m, k \leq m} |f^{(k)}(x)|.$$

This algebra is not normable.

6. The topological dual $\mathcal{E}'(\mathbb{R})$ of $\mathcal{E}(\mathbb{R})$, the so called space of compactly supported distributions. There the multiplication is the convolution, which is defined for nice enough f, g by

$$(f, g) \mapsto f * g, \quad (f * g)(x) := \int_{-\infty}^{\infty} f(x - y)\ g(y)\ \mathrm{d}y.$$

The unit element of $\mathcal{E}(\mathbb{R})$ is the Dirac delta distribution δ_0 at the origin $0 \in \mathbb{R}$. This is a commutative topological algebra with the weak*-topology, but it is not metrizable.

7. Convolution algebras of compactly supported distributions on Lie groups are non-metrizable topological algebras; such an algebra is commutative if and only if the group is commutative.

Remark. Let $\mathcal{A}$ be a topological algebra, $U \subset \mathcal{A}$ open, and $S \subset \mathcal{A}$. Due to the continuity of $((\lambda, x) \mapsto \lambda x) : \mathbb{C} \times \mathcal{A} \to \mathcal{A}$ the set $\lambda U = \{\lambda u \mid u \in U\}$ is open if $\lambda \neq 0$. Due to the continuity of $((x, y) \mapsto x + y) : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ the set $U + S = \{u + s \mid u \in U,\ s \in S\}$ is open.

Exercise. Topological algebras are Hausdorff spaces.

Remark. Notice that in the previous exercise you actually need only the continuities of the mappings $(x, y) \mapsto x + y$ and $x \mapsto -x$, and the fact that $\{0\}$ is a closed set. Indeed, the commutativity of the addition operation is not needed, so that you can actually prove a proposition "Topological groups are Hausdorff spaces"!

Exercise*. Let $\mathcal{A}$ be an algebra and a normed space. Prove that it is a topological algebra if and only if there exists a constant $C < \infty$ such that

$$\|xy\| \leq C \ \|x\| \ \|y\|$$

for every $x, y \in \mathcal{A}$.

Closed ideals

In topological algebras, the good ideals are the closed ones.

Examples. Let $\mathcal{A}$ be a topological algebra; then $\{0\} \subset \mathcal{A}$ is a closed ideal. Let $\mathcal{B}$ be another topological algebra, and $\phi \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ be **continuous**. Then it is easy to see that $\mathrm{Ker}(\phi) = \phi^{-1}(\{0\}) \subset \mathcal{A}$ is a **closed** ideal; this is actually a canonical example of closed ideals.

Proposition. *Let $\mathcal{A}$ be a topological algebra and $\mathcal{J}$ its ideal. Then either $\overline{\mathcal{J}} = \mathcal{A}$ or $\overline{\mathcal{J}} \subset \mathcal{A}$ is a closed ideal.*

Proof. Let $\lambda \in \mathbb{C}$, $x, y \in \overline{\mathcal{J}}$, and $z \in \mathcal{A}$. Take $V \in \mathcal{V}(\lambda x)$. Then there exists $U \in \mathcal{V}(x)$ such that $\lambda U \subset V$ (due to the continuity of the multiplication by a scalar). Since $x \in \overline{\mathcal{J}}$, we may pick $x_0 \in \mathcal{J} \cap U$. Now

$$\lambda x_0 \in \mathcal{J} \cap (\lambda U) \subset \mathcal{J} \cap V,$$

which proves that $\lambda x \in \overline{\mathcal{J}}$. Next take $W \in \mathcal{V}(x+y)$. Then for some $U \in \mathcal{V}(x)$ and $V \in \mathcal{V}(y)$ we have $U + V \subset W$ (due to the continuity of the mapping $(x, y) \mapsto x + y$). Since $x, y \in \overline{\mathcal{J}}$, we may pick $x_0 \in \mathcal{J} \cap U$ and $y_0 \in \mathcal{J} \cap V$. Now

$$x + y \in \mathcal{J} \cap (U + V) \subset \mathcal{J} \cap W,$$

which proves that $x + y \in \overline{\mathcal{J}}$. Finally, we should show that $xz, zx \in \overline{\mathcal{J}}$, but this proof is so similar to the previous steps that it is left for the reader as an easy task □

Topology for quotient algebra. Let $\mathcal{J}$ be an ideal of a topological algebra $\mathcal{A}$. Let τ be the topology of $\mathcal{A}$. For $x \in \mathcal{A}$, define $[x] = x + \mathcal{J}$, and let $[S] = \{[x] \mid x \in S\}$. Then it is easy to check that $\{[U] \mid U \in \tau\}$ is a topology of the quotient algebra $\mathcal{A}/\mathcal{J}$; it is called the *quotient topology.*

Remark. Let $\mathcal{A}$ be a topological algebra and $\mathcal{J}$ be its ideal. The quotient map $(x \mapsto [x]) \in \mathrm{Hom}(\mathcal{A}, \mathcal{A}/\mathcal{J})$ is continuous: namely, if $x \in \mathcal{A}$ and $[V] \in \mathcal{V}_{\mathcal{A}/\mathcal{J}}([x])$ for some $V \in \tau$ then $U := V + \mathcal{J} \in \mathcal{V}(x)$ and $[U] = [V]$.

Lemma. *Let $\mathcal{J}$ be an ideal of a topological algebra $\mathcal{A}$. Then the algebra operations on the quotient algebra $\mathcal{A}/\mathcal{J}$ are continuous.*

Proof. Let us check the continuity of the multiplication in the quotient algebra: Suppose $[x][y] = [xy] \in [W]$, where $W \subset \mathcal{A}$ is an open set (recall that every open set in the quotient algebra is of the form $[W]$). Then

$$xy \in W + \mathcal{J}.$$

Since $\mathcal{A}$ is a topological algebra, there are open sets $U \in \mathcal{V}_{\mathcal{A}}(x)$ and $V \in \mathcal{V}_{\mathcal{A}}(y)$ satisfying

$$UV \subset W + \mathcal{J}.$$

Now $[U] \in \mathcal{V}_{\mathcal{A}/\mathcal{J}}([x])$ and $[V] \in \mathcal{V}_{\mathcal{A}/\mathcal{J}}([y])$. Furthermore, $[U][V] \subset [W]$ because

$$(U + \mathcal{J})(V + \mathcal{J}) \subset UV + \mathcal{J} \subset W + \mathcal{J};$$

we have proven the continuity of the multiplication $([x], [y]) \mapsto [x][y]$. As an easy exercise, we leave it for the reader to verify the continuities of the mappings $(\lambda, [x]) \mapsto \lambda[x]$ and $([x], [y]) \mapsto [x] + [y]$ $\square$

Exercise. Complete the previous proof by showing the continuities of the mappings $(\lambda, [x]) \mapsto \lambda[x]$ and $([x], [y]) \mapsto [x] + [y]$.

With the previous Lemma, we conclude:

Proposition. *Let $\mathcal{J}$ be an ideal of a topological algebra $\mathcal{A}$. Then $\mathcal{A}/\mathcal{J}$ is a topological algebra if and only if $\mathcal{J}$ is closed.*

Proof. If the quotient algebra is a topological algebra then $\{[0]\} = \{\mathcal{J}\}$ is a closed subset of $\mathcal{A}/\mathcal{J}$; since the quotient homomorphism is a continuous mapping, $\mathcal{J} = \mathrm{Ker}(x \mapsto [x]) \subset \mathcal{A}$ must be a closed set.

Conversely, suppose $\mathcal{J}$ is a closed ideal of a topological algebra $\mathcal{A}$. Then we deduce that

$$(\mathcal{A}/\mathcal{J}) \setminus \{[0]\} = [\mathcal{A} \setminus \mathcal{J}]$$

is an open subset of the quotient algebra, so that $\{[0]\} \subset \mathcal{A}/\mathcal{J}$ is closed $\square$

Remark. Let X be a topological vector space and M be its subspace. The reader should be able to define the *quotient topology* for the quotient vector space $X/M = \{[x] := x + M \mid x \in X\}$. Now X/M is a topological vector space if and only if M is a closed subspace.

Let $M \subset X$ be a closed subspace. If d is a metric on X then there is a natural metric for X/M:

$$([x],[y]) \mapsto d([x],[y]) := \inf_{z \in M} d(x-y,z),$$

and if X is a complete metric space then X/M is also complete. Moreover, if $x \mapsto \|x\|$ is a norm on X then there is a natural norm for X/M:

$$[x] \mapsto \|[x]\| := \inf_{z \in M} \|x - z\|.$$

5 Compact spaces

In this section we mainly concentrate on compact Hausdorff spaces, though some results deal with more general classes of topological spaces. Roughly, Hausdorff spaces have enough open sets to distinguish between any two points, while compact spaces "do not have too many open sets". Combining these two properties, compact Hausdorff spaces form an extremely beautiful class to study.

Compact space. Let X be a set and $K \subset X$. A family $\mathcal{S} \subset \mathcal{P}(X)$ is called a *cover of* K if

$$K \subset \bigcup \mathcal{S};$$

if the cover $\mathcal{S}$ is a finite set, it is called a *finite cover.* A cover $\mathcal{S}$ of $K \subset X$ has a *subcover* $\mathcal{S}' \subset \mathcal{S}$ if $\mathcal{S}'$ itself is a cover of K.

Let (X, τ) be a topological space. An *open cover of* X is a cover $\mathcal{U} \subset \tau$ of X. A subset $K \subset X$ is *compact* (more precisely τ*-compact*) if every open cover of K has a finite subcover, i.e.

$$\forall \mathcal{U} \subset \tau \ \exists \mathcal{U}' \subset \mathcal{U} : \ K \subset \bigcup \mathcal{U} \Rightarrow K \subset \bigcup \mathcal{U}' \quad \text{and} \quad |\mathcal{U}'| < \infty.$$

We say that (X, τ) is a compact space if X itself is τ-compact.

Examples.

1. If τ_1 and τ_2 are topologies of X, $\tau_1 \subset \tau_2$, and (X, τ_2) is a compact space then (X, τ_1) is a compact space.

2. $(X, \{\emptyset, X\})$ is a compact space.

3. If $|X| = \infty$ then $(X, \mathcal{P}(X))$ is not a compact space. Clearly any space with a finite topology is compact. Even though a compact topology can be of *any* cardinality, it is in a sense "not far away from being finite".

4. A metric space is compact if and only if it is sequentially compact (i.e. every sequence contains a converging subsequence).

5. A subset $X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded (Heine–Borel Theorem).

6. A theorem due to Frigyes Riesz asserts that a closed ball in a normed vector space over $\mathbb{C}$ (or $\mathbb{R}$) is compact if and only if the vector space is finite-dimensional.

Exercise. A union of two compact sets is compact.

Proposition. *An intersection of a compact set and a closed set is compact.*

Proof. Let $K \subset X$ be a compact set, and $C \subset X$ be a closed set. Let $\mathcal{U}$ be an open cover of $K \cap C$. Then $\{X \setminus C\} \cup \mathcal{U}$ is an open cover of K, thus having a finite subcover $\mathcal{U}'$. Then $\mathcal{U}' \setminus \{X \setminus C\} \subset \mathcal{U}$ is a finite subcover of $K \cap C$; hence $K \cap C$ is compact □

Proposition. *Let X be a compact space and $f : X \to Y$ continuous. Then $f(X) \subset Y$ is compact.*

Proof. Let $\mathcal{V}$ be an open cover of $f(X)$. Then $\mathcal{U} := \{f^{-1}(V) \mid V \in \mathcal{V}\}$ is an open cover of X, thus having a finite subcover $\mathcal{U}'$. Hence $f(X)$ is covered by $\{f(U) \mid U \in \mathcal{U}'\} \subset \mathcal{V}$ □

Corollary. *If X is compact and $f \in C(X)$ then $|f|$ attains its greatest value on X (here $|f|(x) := |f(x)|$)* □

5.1 Compact Hausdorff spaces

Theorem. *Let X be a Hausdorff space, $A, B \subset X$ compact subsets, and $A \cap B = \emptyset$. Then there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. (In particular, compact sets in a Hausdorff space are closed.)*

Proof. The proof is trivial if $A = \emptyset$ or $B = \emptyset$. So assume $x \in A$ and $y \in B$. Since X is a Hausdorff space and $x \neq y$, we can choose neighborhoods $U_{xy} \in \mathcal{V}(x)$ and $V_{xy} \in \mathcal{V}(y)$ such that $U_{xy} \cap V_{xy} = \emptyset$. The collection $\mathcal{P} = \{V_{xy} \mid y \in B\}$ is an open cover of the compact set B, so that it has a finite subcover

$$\mathcal{P}_x = \{V_{xy_j} \mid 1 \leq j \leq n_x\} \subset \mathcal{P}$$

for some $n_x \in \mathbb{N}$. Let

$$U_x := \bigcap_{j=1}^{n_x} U_{xy_j}.$$

Now $\mathcal{O} = \{U_x \mid x \in A\}$ is an open cover of the compact set A, so that it has a finite subcover

$$\mathcal{O}' = \{U_{x_i} \mid 1 \leq i \leq m\} \subset \mathcal{O}.$$

Then define

$$U := \bigcup \mathcal{O}', \quad V := \bigcap_{i=1}^{m} \bigcup \mathcal{P}_{x_i}.$$

It is an easy task to check that U and V have desired properties $\square$

Corollary. *Let X be a compact Hausdorff space, $x \in X$, and $W \in \mathcal{V}(x)$. Then there exists $U \in \mathcal{V}(x)$ such that $\overline{U} \subset W$.*

Proof. Now $\{x\}$ and $X \setminus W$ are closed sets in a compact space, thus they are compact. Since these sets are disjoint, there exist open disjoint sets $U, V \subset X$ such that $x \in U$ and $X \setminus W \subset V$; i.e.

$$x \in U \subset X \setminus V \subset W.$$

Hence $x \in U \subset \overline{U} \subset X \setminus V \subset W$ $\square$

Proposition. *Let (X, τ_X) be a compact space and (Y, τ_Y) a Hausdorff space. A bijective continuous mapping $f : X \to Y$ is a homeomorphism.*

Proof. Let $U \in \tau_X$. Then $X \setminus U$ is closed, hence compact. Consequently, $f(X \setminus U)$ is compact, and due to the Hausdorff property $f(X \setminus U)$ is closed. Therefore $(f^{-1})^{-1}(U) = f(U)$ is open $\square$

Corollary. *Let X be a set with a compact topology τ_2 and a Hausdorff topology τ_1. If $\tau_1 \subset \tau_2$ then $\tau_1 = \tau_2$.*

Proof. The identity mapping $(x \mapsto x) : X \to X$ is a continuous bijection from (X, τ_2) to (X, τ_1) $\square$

A more direct proof of the Corollary. Let $U \in \tau_2$. Since (X, τ_2) is compact and $X \setminus U$ is τ_2-closed, $X \setminus U$ must be τ_2-compact. Now $\tau_1 \subset \tau_2$, so that $X \setminus U$ is τ_1-compact. (X, τ_1) is Hausdorff, implying that $X \setminus U$ is τ_1-closed, thus $U \in \tau_1$; this yields $\tau_2 \subset \tau_1$ $\square$

Functional separation

A family $\mathcal{F}$ of mappings $X \to \mathbb{C}$ is said to *separate the points of the set X* if there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$ whenever $x \neq y$. Later in these notes we shall discover that a compact space X is metrizable if and only if

$C(X)$ is separable and separates the points of X.

Urysohn's Lemma is the key result of this section:

Urysohn's Lemma (1923?). *Let X be a compact Hausdorff space, $A, B \subset X$ closed non-empty sets, $A \cap B = \emptyset$. Then there exists $f \in C(X)$ such that*

$$0 \leq f \leq 1, \quad f(A) = \{0\}, \quad f(B) = \{1\}.$$

Proof. The set $\mathbb{Q} \cap [0,1]$ is countably infinite; let $\phi : \mathbb{N} \to \mathbb{Q} \cap [0,1]$ be a bijection satisfying $\phi(0) = 0$ and $\phi(1) = 1$. Choose open sets $U_0, U_1 \subset X$ such that

$$A \subset U_0 \subset \overline{U_0} \subset U_1 \subset \overline{U_1} \subset X \setminus B.$$

Then we proceed inductively as follows: Suppose we have chosen open sets $U_{\phi(0)}, U_{\phi(1)}, \ldots, U_{\phi(n)}$ such that

$$\phi(i) < \phi(j) \Rightarrow \overline{U_{\phi(i)}} \subset U_{\phi(j)}.$$

Let us choose an open set $U_{\phi(n+1)} \subset X$ such that

$$\phi(i) < \phi(n+1) < \phi(j) \Rightarrow \overline{U_{\phi(i)}} \subset U_{\phi(n+1)} \subset \overline{U_{\phi(n+1)}} \subset U_{\phi(j)}$$

whenever $0 \leq i, j \leq n$. Let us define

$$r < 0 \Rightarrow U_r := \emptyset, \quad s > 1 \Rightarrow U_s := X.$$

Hence for each $q \in \mathbb{Q}$ we get an open set $U_q \subset X$ such that

$$\forall r, s \in \mathbb{Q} : \; r < s \Rightarrow \overline{U_r} \subset U_s.$$

Let us define a function $f : X \to [0,1]$ by

$$f(x) := \inf\{r : \; x \in U_r\}.$$

Clearly $0 \leq f \leq 1$, $f(A) = \{0\}$ and $f(B) = \{1\}$.

Let us prove that f is continuous. Take $x \in X$ and $\varepsilon > 0$. Take $r, s \in \mathbb{Q}$ such that

$$f(x) - \varepsilon < r < f(x) < s < f(x) + \varepsilon;$$

then f is continuous at x, since $x \in U_s \setminus \overline{U_r}$ and for every $y \in U_s \setminus \overline{U_r}$ we have $|f(y) - f(x)| < \varepsilon$. Thus $f \in C(X)$ □

Corollary. *Let X be a compact space. Then $C(X)$ separates the points of X if and only if X is Hausdorff.*

Exercise. Prove the previous Corollary.

Appendix on complex analysis

Let $\Omega \subset \mathbb{C}$ be open. A function $f : \Omega \to \mathbb{C}$ is called *holomorphic in* Ω, denoted by $f \in H(\Omega)$, if the limit

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for every $z \in \Omega$. Then Cauchy's integral formula provides a power series representation

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

converging uniformly on the compact subsets of the disk

$$\mathbb{D}(a,r) = \{z \in \mathbb{C} : \ |z-a| < r\} \subset \Omega;$$

here $c_n = f^{(n)}(a)/n!$, where $f^{(0)} = f$ and $f^{(n+1)} = f^{(n)\prime}$.

Liouville's Theorem. *Let $f \in H(\mathbb{C})$ such that $|f|$ is bounded. Then f is constant, i.e. $f(z) \equiv f(0)$ for every $z \in \mathbb{C}$.*

Proof. Since $f \in H(\mathbb{C})$, we have a power series representation

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

converging uniformly on the compact sets in the complex plane. Thereby

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\phi})|^2 \, \mathrm{d}\phi &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n,m} c_n \, \overline{c_m} \, r^{n+m} \, e^{i(n-m)\phi} \, \mathrm{d}\phi \\
&= \sum_{n,m} c_n \, \overline{c_m} \, r^{n+m} \, \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\phi} \, \mathrm{d}\phi \\
&= \sum_{n=0}^{\infty} |c_n|^2 r^{2n}
\end{aligned}$$

for every $r > 0$. Hence the fact

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\phi})|^2 \, \mathrm{d}\phi \le \sup_{z \in \mathbb{C}} |f(z)|^2 < \infty$$

implies $c_n = 0$ for every $n \ge 1$; thus $f(z) \equiv c_0 = f(0)$ $\square$

Appendix on functional analysis

Let X, Y be normed spaces with norms $x \mapsto \|x\|_X$ and $y \mapsto \|y\|_Y$, respectively. The set of bounded linear mappings $X \to Y$ is denoted by $\mathcal{L}(X,Y)$; the operator norm $(A \mapsto \|A\|) : \mathcal{L}(X,Y) \to \mathbb{R}$ is defined by

$$\|A\| := \sup_{x \in X: \|x\|_X \leq 1} \|Ax\|_Y.$$

Let us denote the dual of a normed space X by $X' := \mathcal{L}(X, \mathbb{C})$.

Hahn–Banach Theorem. *Let X be a normed vector space, $M \subset X$ be a vector subspace, and $f : M \to \mathbb{C}$ a bounded linear functional. Then there is a bounded linear functional $F : X \to \mathbb{C}$ such that $\|f\| = \|F\|$ and $f(x) = F(x)$ for every $x \in M$* □

Corollary. *Let X is a normed space. Then*

$$\|x\| = \max_{F \in X': \|F\| \leq 1} |F(x)|$$

for every $x \in X$ □

Banach–Steinhaus Theorem (Uniform Boundedness Principle). *Let X, Y be Banach spaces and $\{T_j\}_{j \in J} \subset \mathcal{L}(X,Y)$. If*

$$\sup_{j \in J} \|T_j x\| < \infty$$

for every $x \in X$ then $\sup_{j \in J} \|T_j\| < \infty$ □

6 Banach algebras

Banach algebra. An algebra $\mathcal{A}$ is called a *Banach algebra* if it is a Banach space satisfying

$$\|xy\| \leq \|x\| \, \|y\|$$

for every $x, y \in \mathcal{A}$ and

$$\|\mathbb{I}\| = 1.$$

The next exercise is very important:

Exercise*. *Let K be a compact space. Show that $C(K)$ is a Banach algebra with the norm $f \mapsto \|f\| = \max_{x \in K} |f(x)|$.*

Examples. Let X be a Banach space. Then the Banach space $\mathcal{L}(X)$ of bounded linear operators $X \to X$ is a Banach algebra, when the multiplication is the composition of operators, since

$$\|AB\| \leq \|A\| \, \|B\|$$

for every $A, B \in \mathcal{L}(X)$; the unit is the identity operator $I : X \to X$, $x \mapsto x$. Actually, this is not far away from characterizing all the Banach algebras:

Theorem. *A Banach algebra $\mathcal{A}$ is isometrically isomorphic to a norm closed subalgebra of $\mathcal{L}(X)$ for a Banach space X.*

Proof. Here $X := \mathcal{A}$. For $x \in \mathcal{A}$, let us define

$$m(x) : \mathcal{A} \to \mathcal{A} \quad \text{by} \quad m(x)y := xy.$$

Obviously $m(x)$ is a linear mapping, $m(xy) = m(x)m(y)$, $m(\mathbb{I}_{\mathcal{A}}) = \mathbb{I}_{\mathcal{L}(\mathcal{A})}$, and

$$\begin{aligned} \|m(x)\| &= \sup_{y \in \mathcal{A}:\ \|y\| \leq 1} \|xy\| \\ &\leq \sup_{y \in \mathcal{A}:\ \|y\| \leq 1} (\|x\| \, \|y\|) = \|x\| = \|m(x)\mathbb{I}_{\mathcal{A}}\| \\ &\leq \|m(x)\| \, \|\mathbb{I}_{\mathcal{A}}\| = \|m(x)\|; \end{aligned}$$

briefly, $m = (x \mapsto m(x)) \in \mathrm{Hom}(\mathcal{A}, \mathcal{L}(\mathcal{A}))$ is isometric. Thereby $m(\mathcal{A}) \subset \mathcal{L}(\mathcal{A})$ is a closed subspace and a Banach algebra □

Exercise*. Let a Banach space $\mathcal{A}$ be a topological algebra. Equip $\mathcal{A}$ with an equivalent Banach algebra norm.

Exercise. Let $\mathcal{A}$ be a Banach algebra, and let $x, y \in \mathcal{A}$ satisfy

$$x^2 = x, \quad y^2 = y, \quad xy = yx.$$

Show that either $x = y$ or $\|x - y\| \geq 1$. Give an example of a Banach algebra $\mathcal{A}$ with elements $x, y \in \mathcal{A}$ such that $x^2 = x \neq y = y^2$ and $\|x - y\| < 1$.

Proposition. *Let $\mathcal{A}$ be a Banach algebra. Then* $\mathrm{Hom}(\mathcal{A}, \mathbb{C}) \subset \mathcal{A}'$ *and* $\|\phi\| = 1$ *for every* $\phi \in \mathrm{Hom}(\mathcal{A}, \mathbb{C})$.

Proof. Let $x \in \mathcal{A}$, $\|x\| < 1$. Let

$$y_n := \sum_{j=0}^{n} x^j,$$

where $x^0 := \mathbb{I}$. If $n > m$ then

$$\begin{aligned} \|y_n - y_m\| &= \|x^m + x^{m+1} + \ldots + x^n\| \\ &\leq \|x\|^m \left(1 + \|x\| + \ldots + \|x\|^{n-m}\right) \\ &= \|x\|^m \frac{1 - \|x\|^{n-m+1}}{1 - \|x\|} \to_{n>m\to\infty} 0; \end{aligned}$$

thus $(y_n)_{n=1}^{\infty} \subset \mathcal{A}$ is a Cauchy sequence. There exists $y = \lim_{n\to\infty} y_n \in \mathcal{A}$, because $\mathcal{A}$ is complete. Since $x^n \to 0$ and

$$y_n(\mathbb{I} - x) = \mathbb{I} - x^{n+1} = (\mathbb{I} - x)y_n,$$

we deduce $y = (\mathbb{I} - x)^{-1}$. Suppose $\lambda = \phi(x)$, $|\lambda| > \|x\|$; now $\|\lambda^{-1}x\| = |\lambda|^{-1}\, \|x\| < 1$, so that $\mathbb{I} - \lambda^{-1}x$ is invertible. Then

$$\begin{aligned} 1 &= \phi(\mathbb{I}) = \phi\left((\mathbb{I} - \lambda^{-1}x)(\mathbb{I} - \lambda^{-1}x)^{-1}\right) \\ &= \phi\left(\mathbb{I} - \lambda^{-1}x\right)\ \phi\left((\mathbb{I} - \lambda^{-1}x)^{-1}\right) \\ &= (1 - \lambda^{-1}\phi(x))\ \phi\left((\mathbb{I} - \lambda^{-1}x)^{-1}\right) = 0, \end{aligned}$$

a contradiction; hence

$$\forall x \in \mathcal{A}: \ |\phi(x)| \leq \|x\|,$$

that is $\|\phi\| \leq 1$. Finally, $\phi(\mathbb{I}) = 1$, so that $\|\phi\| = 1$ □

Lemma. *Let $\mathcal{A}$ be a Banach algebra. The set $G(\mathcal{A}) \subset \mathcal{A}$ of its invertible elements is open. The mapping $(x \mapsto x^{-1}) : G(\mathcal{A}) \to G(\mathcal{A})$ is a homeomorphism.*

Proof. Take $x \in G(\mathcal{A})$ and $h \in \mathcal{A}$. As in the proof of the previous Proposition, we see that

$$x - h = x(\mathbb{I} - x^{-1}h)$$

is invertible if $\|x^{-1}\| \, \|h\| < 1$, that is $\|h\| < \|x^{-1}\|^{-1}$; thus $G(\mathcal{A}) \subset \mathcal{A}$ is open.

The mapping $x \mapsto x^{-1}$ is clearly its own inverse. Moreover

$$\begin{aligned} \|(x-h)^{-1} - x^{-1}\| &= \|(\mathbb{I} - x^{-1}h)^{-1}x^{-1} - x^{-1}\| \\ &\leq \|(\mathbb{I} - x^{-1}h)^{-1} - \mathbb{I}\| \, \|x^{-1}\| = \|\sum_{n=1}^{\infty}(x^{-1}h)^n\| \, \|x^{-1}\| \\ &\leq \|h\| \left(\sum_{n=1}^{\infty} \|x^{-1}\|^{n+1} \, \|h\|^{n-1} \right) \to_{h \to 0} 0; \end{aligned}$$

hence $x \mapsto x^{-1}$ is a homeomorphism □

Exercise. Let $\mathcal{A}$ be a Banach algebra. We say that $x \in \mathcal{A}$ is a *topological zero divisor* if there exists a sequence $(y_n)_{n=1}^{\infty} \subset \mathcal{A}$ such that $\|y_n\| = 1$ for all n and

$$\lim_{n\to\infty} xy_n = 0 = \lim_{n\to\infty} y_n x.$$

(a) Show that if $(x_n)_{n=1}^{\infty} \subset G(\mathcal{A})$ satisfies $x_n \to x \in \partial G(\mathcal{A})$ then $\|x_n^{-1}\| \to \infty$.
(b) Using this result, show that the boundary points of $G(\mathcal{A})$ are topological zero divisors.
(c) In what kind of Banach algebras 0 is the only topological zero divisor?

Theorem (Gelfand, 1939). *Let $\mathcal{A}$ be a Banach algebra and $x \in \mathcal{A}$. The spectrum $\sigma(x) \subset \mathbb{C}$ is a non-empty compact set.*

Proof. Let $x \in \mathcal{A}$. Then $\sigma(x)$ belongs to a 0-centered disc of radius $\|x\|$ in the complex plane: for if $\lambda \in \mathbb{C}$, $|\lambda| > \|x\|$ then $\mathbb{I} - \lambda^{-1}x$ is invertible, equivalently $\lambda\mathbb{I} - x$ is invertible.

The mapping $g : \mathbb{C} \to \mathcal{A}$, $\lambda \mapsto \lambda\mathbb{I} - x$, is continuous; the set $G(\mathcal{A}) \subset \mathcal{A}$ of invertible elements is open, so that

$$\mathbb{C} \setminus \sigma(x) = g^{-1}(G(\mathcal{A}))$$

is open. Thus $\sigma(x) \in \mathbb{C}$ is closed and bounded, i.e. compact by Heine–Borel.

The hard part is to prove the non-emptiness of the spectrum. Let us define the *resolvent mapping* $R : \mathbb{C} \setminus \sigma(x) \to G(\mathcal{A})$ by

$$R(\lambda) = (\lambda \mathbb{I} - x)^{-1}.$$

We know that this mapping is continuous, because it is composed of continuous mappings

$$(\lambda \mapsto \lambda \mathbb{I} - x) : \mathbb{C} \setminus \sigma(x) \to G(\mathcal{A}) \quad \text{and} \quad (y \mapsto y^{-1}) : G(\mathcal{A}) \to G(\mathcal{A}).$$

We want to show that R is *weakly holomorphic*, that is $f \circ R \in H(\mathbb{C} \setminus \sigma(x))$ for every $f \in \mathcal{A}' = \mathcal{L}(\mathcal{A}, \mathbb{C})$. Let $z \in \mathbb{C} \setminus \sigma(x)$, $f \in \mathcal{A}'$. Then we calculate

$$\begin{aligned}
\frac{(f \circ R)(z+h) - (f \circ R)(z)}{h} &= f\left(\frac{R(z+h) - R(z)}{h}\right) \\
&= f\left(\frac{R(z+h)R(z)^{-1} - \mathbb{I}}{h} \, R(z)\right) \\
&= f\left(\frac{R(z+h)(R(z+h)^{-1} - h\mathbb{I}) - \mathbb{I}}{h} \, R(z)\right) \\
&= f(-R(z+h)R(z)) \\
&\to_{h \to 0} f(-R(z)^2),
\end{aligned}$$

because f and R are continuous; thus R is weakly holomorphic.

Suppose $|\lambda| > \|x\|$. Then

$$\begin{aligned}
\|R(\lambda)\| &= \|(\lambda \mathbb{I} - x)^{-1}\| = |\lambda|^{-1} \, \|(\mathbb{I} - x/\lambda)^{-1}\| = |\lambda|^{-1} \left\| \sum_{j=0}^{\infty} (x/\lambda)^j \right\| \\
&\leq |\lambda|^{-1} \sum_{j=0}^{\infty} \|x/\lambda\|^{-j} = |\lambda|^{-1} \frac{1}{1 - \|x/\lambda\|} = \frac{1}{|\lambda| - \|x\|} \\
&\to_{|\lambda| \to \infty} 0.
\end{aligned}$$

Thereby

$$(f \circ R)(\lambda) \to_{|\lambda| \to \infty} 0$$

for every $f \in \mathcal{A}'$. To get a contradiction, suppose $\sigma(x) = \emptyset$. Then $f \circ R \in H(\mathbb{C})$ is 0 by Liouville's Theorem (see Appendix), for every $f \in \mathcal{A}'$; the Hahn-Banach Theorem says that then $R(\lambda) = 0$ for every $\lambda \in \mathbb{C}$; this is a contradiction, since $0 \notin G(\mathcal{A})$. Thus $\sigma(x) \neq \emptyset$ $\square$

Exercise. Let $\mathcal{A}$ be a Banach algebra, $x \in \mathcal{A}$, $\Omega \subset \mathbb{C}$ an open set, and $\sigma(x) \subset \Omega$. Then

$$\exists \delta > 0 \ \forall y \in \mathcal{A} : \ \|y\| < \delta \Rightarrow \sigma(x+y) \subset \Omega.$$

Corollary (Gelfand–Mazur). *Let $\mathcal{A}$ be a Banach algebra where $0 \in \mathcal{A}$ is the only non-invertible element. Then $\mathcal{A}$ is isometrically isomorphic to $\mathbb{C}$.*

Proof. Take $x \in \mathcal{A}$, $x \neq 0$. Since $\sigma(x) \neq \emptyset$, pick $\lambda(x) \in \sigma(x)$. Then $\lambda(x)\mathbb{I} - x$ is non-invertible, so that it must be 0; $x = \lambda(x)\mathbb{I}$. By defining $\lambda(0) = 0$, we have an algebra isomorphism

$$\lambda : \mathcal{A} \to \mathbb{C}.$$

Moreover, $|\lambda(x)| = \|\lambda(x)\mathbb{I}\| = \|x\|$ □

Exercise. Let $\mathcal{A}$ be a Banach algebra, and suppose that there exists $C < \infty$ such that

$$\|x\| \ \|y\| \leq C \ \|xy\|$$

for every $x, y \in \mathcal{A}$. Show that $\mathcal{A} \cong \mathbb{C}$ isometrically.

Spectral radius. Let $\mathcal{A}$ be a Banach algebra. The *spectral radius* of $x \in \mathcal{A}$ is

$$\rho(x) := \sup_{\lambda \in \sigma(x)} |\lambda|;$$

this is well-defined, because due to Gelfand the spectrum in non-empty. In other words, $\overline{\mathbb{D}(0, \rho(x))} = \{\lambda \in \mathbb{C} : \ |\lambda| \leq \rho(x)\}$ is the smallest 0-centered closed disk containing $\sigma(x) \subset \mathbb{C}$. Notice that $\rho(x) \leq \|x\|$, since $\lambda\mathbb{I} - x = \lambda(\mathbb{I} - x/\lambda)$ is invertible if $|\lambda| > \|x\|$.

Spectral Radius Formula (Beurling, 1938; Gelfand, 1939). *Let $\mathcal{A}$ be a Banach algebra, $x \in \mathcal{A}$. Then*

$$\rho(x) = \lim_{n \to \infty} \|x^n\|^{1/n}.$$

Proof. For $x = 0$ the claim is trivial, so let us assume that $x \neq 0$. By Gelfand's Theorem, $\sigma(x) \neq \emptyset$. Let $\lambda \in \sigma(x)$ and $n \geq 1$. Notice that in an algebra, if both ab and ba are invertible then the elements a, b are invertible. Therefore

$$\lambda^n \mathbb{I} - x^n = (\lambda\mathbb{I} - x) \left(\sum_{k=0}^{n-1} \lambda^{n-1-k} x^k\right) = \left(\sum_{k=0}^{n-1} \lambda^{n-1-k} x^k\right) (\lambda\mathbb{I} - x)$$

implies that $\lambda^n \in \sigma(x^n)$. Thus $|\lambda^n| \leq \|x^n\|$, so that

$$\rho(x) = \sup_{\lambda\in\sigma(x)} |\lambda| \leq \liminf_{n\to\infty} \|x^n\|^{1/n}.$$

Let $f \in \mathcal{A}'$ and $\lambda \in \mathbb{C}$, $|\lambda| > \|x\|$. Then

$$\begin{aligned} f(R(\lambda)) &= f\left((\lambda\mathbb{I} - x)^{-1}\right) = f\left(\lambda^{-1}(\mathbb{I} - \lambda^{-1}x)^{-1}\right) \\ &= f\left(\lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} x^n\right) \\ &= \lambda^{-1} \sum_{n=0}^{\infty} f(\lambda^{-n} x^n). \end{aligned}$$

This formula is true also when $|\lambda| > \rho(x)$, because $f \circ R$ is holomorphic in $\mathbb{C} \setminus \sigma(x) \supset \mathbb{C} \setminus \overline{\mathbb{D}(0, \rho(x))}$. Hence if we define $T_{\lambda,x,n} \in \mathcal{A}'' = \mathcal{L}(\mathcal{A}', \mathbb{C})$ by $T_{\lambda,x,n}(f) := f(\lambda^{-n}x^n)$, we obtain

$$\sup_{n\in\mathbb{N}} |T_{\lambda,x,n}(f)| = \sup_{n\in\mathbb{N}} |f(\lambda^{-n}x^n)| < \infty \qquad (\text{when } |\lambda| > \rho(x))$$

for every $f \in \mathcal{A}'$; the Banach–Steinhaus Theorem applied on the family $\{T_{\lambda,x,n}\}_{n\in\mathbb{N}}$ shows that

$$M_{\lambda,x} := \sup_{n\in\mathbb{N}} \|T_{\lambda,x,n}\| < \infty,$$

so that we have

$$\begin{aligned} \|\lambda^{-n}x^n\| &\overset{\text{Hahn–Banach}}{=} \sup_{f\in\mathcal{A}':\|f\|\leq 1} |f(\lambda^{-n}x^n)| \\ &= \sup_{f\in\mathcal{A}':\|f\|\leq 1} |T_{\lambda,x,n}(f)| \\ &= \|T_{\lambda,x,n}\| \\ &\leq M_{\lambda,x}. \end{aligned}$$

Hence

$$\|x^n\|^{1/n} \leq M_{\lambda,x}^{1/n} |\lambda| \to_{n\to\infty} |\lambda|,$$

when $|\lambda| > \rho(x)$. Thus

$$\limsup_{n\to\infty} \|x^n\|^{1/n} \le \rho(x);$$

collecting the results, the Spectral Radius Formula is verified □

Remark 1. The Spectral Radius Formula contains startling information: the spectral radius $\rho(x)$ is purely an algebraic property (though related to a topological algebra), but the quantity $\lim \|x^n\|^{1/n}$ relies on both algebraic and metric properties! Yet the results are equal!

Remark 2. $\rho(x)^{-1}$ is the radius of convergence of the $\mathcal{A}$-valued power series $\lambda \mapsto \sum_{n=0}^{\infty} \lambda^n x^n$.

Remark 3. Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ its Banach subalgebra. If $x \in \mathcal{B}$ then

$$\sigma_{\mathcal{A}}(x) \subset \sigma_{\mathcal{B}}(x)$$

and the inclusion can be proper, but the spectral radii for both Banach algebras are the same, since

$$\rho_{\mathcal{A}}(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \rho_{\mathcal{B}}(x).$$

Exercise. Let $\mathcal{A}$ be a Banach algebra, $x, y \in \mathcal{A}$. Show that $\rho(xy) = \rho(yx)$. Show that if $x \in \mathcal{A}$ is *nilpotent* (i.e. $x^k = 0$ for some $k \in \mathbb{N}$) then $\sigma(x) = \{0\}$. Give examples of nilpotent linear operators.

Exercise. Let $\mathcal{A}$ be a Banach algebra and $x, y \in \mathcal{A}$ such that $xy = yx$. Prove that $\rho(xy) \le \rho(x)\rho(y)$.

7 New topologies from old ones

In this section families of mappings transfer (induce and co-induce) topologies from topological spaces to a set in natural ways. The most important cases for us are quotient and product spaces.

Comparison of topologies. If (X, τ_1) and (X, τ_2) are topological spaces and $\tau_1 \subset \tau_2$, we say that τ_1 is *weaker* than τ_2 and τ_2 is *stronger* than τ_1.

7.1 Co-induction

Co-induced topology. Let X and J be sets, (X_j, τ_j) be topological spaces for every $j \in J$, and $\mathcal{F} = \{f_j : X_j \to X \mid j \in J\}$ be a family mappings. The *$\mathcal{F}$-co-induced topology* of X is the strongest topology τ on X such that the mappings f_j are continuous for every $j \in J$. Indeed, this definition is sound, because

$$\tau = \{U \subset X \mid \forall j \in J : \ f_j^{-1}(U) \in \tau_j\},$$

as the reader may easily verify.

Example. Let $\mathcal{A}$ be a topological vector space and $\mathcal{J}$ its subspace. Let us denote $[x] := x + \mathcal{J}$ for $x \in \mathcal{A}$. Then the quotient topology of $\mathcal{A}/\mathcal{J} = \{[x] \mid x \in \mathcal{A}\}$ is the $\{(x \mapsto [x]) : \mathcal{A} \to \mathcal{A}/\mathcal{J}\}$-co-induced topology.

Example. Let (X, τ_X) be a topological space. Let $R \subset X \times X$ be an equivalence relation. Let

$$[x] := \{y \in X \mid (x, y) \in R\},$$

$$X/R := \{[x] \mid x \in X\},$$

and define the *quotient map* $p : X \to X/R$ by $x \mapsto [x]$. The *quotient topology* of the *quotient space* X/R is the $\{p\}$-co-induced topology on X/R. Notice that X/R is compact if X is compact, since $p : X \to X/R$ is a continuous surjection.

Remark. The message of the following exercise is that if our compact space X is not Hausdorff, we "factor out" inessential information that $C(X)$ "does not see" to obtain a compact Hausdorff space related nicely to X.

Exercise*. Let X be a topological space, and define $C \subset X \times X$ by

$$(x, y) \in C \stackrel{\text{definition}}{\iff} \forall f \in C(X): \ f(x) = f(y).$$

Prove:
(a) C is an equivalence relation on X.
(b) There is a natural bijection between the sets $C(X)$ and $C(X/C)$.
(c) X/C is a Hausdorff space.
(d) If X is a compact Hausdorff space then $X \cong X/C$.

Exercise. For $A \subset X$ the notation X/A means X/R_A, where the equivalence relation R_A is given by

$$(x, y) \in R_A \stackrel{\text{definition}}{\iff} x = y \text{ or } \{x, y\} \subset A.$$

Let X be a topological space, and let $\infty \subset X$ be a closed subset. Prove that the mapping

$$X \setminus \infty \to (X/\infty) \setminus \{\infty\}, \quad x \mapsto [x],$$

is a homeomorphism.

Finally, let us state a basic property of co-induced topologies:

Proposition. *Let X have the $\mathcal{F}$-co-induced topology, and Y be a topological space. A mapping $g : X \to Y$ is continuous if and only if $g \circ f$ is continuous for every $f \in \mathcal{F}$.*

Proof. If g is continuous then the composed mapping $g \circ f$ is continuous for every $f \in \mathcal{F}$.

Conversely, suppose $g \circ f_j$ is continuous for every $f_j \in \mathcal{F}$, $f_j : X_j \to X$. Let $V \subset Y$ be open. Then

$$f_j^{-1}(g^{-1}(V)) = (g \circ f_j)^{-1}(V) \subset X_j \quad \text{is open;}$$

thereby $g^{-1}(V) = f_j(f_j^{-1}(g^{-1}(V))) \subset X$ is open □

Corollary. *Let X, Y be topological spaces, R be an equivalence relation on X, and endow X/R with the quotient topology. A mapping $f : X/R \to Y$ is continuous if and only if $(x \mapsto f([x])) : X \to Y$ is continuous* □

7.2 Induction

Induced topology. Let X and J be sets, (X_j, τ_j) be topological spaces for every $j \in J$ and $\mathcal{F} = \{f_j : X \to X_j \mid j \in J\}$ be a family of mappings. The *$\mathcal{F}$-induced topology* of X is the weakest topology τ on X such that the mappings f_j are continuous for every $j \in J$.

Example. Let (X, τ_X) be a topological space, $A \subset X$, and let $\iota : A \to X$ be defined by $\iota(a) = a$. Then the $\{\iota\}$-induced topology on A is

$$\tau_X|_A := \{U \cap A \mid U \in \tau_X\}.$$

This is called the *relative topology* of A. Let $f : X \to Y$. The restriction $f|_A = f \circ \iota : A \to Y$ satisfies $f|_A(a) = f(a)$ for every $a \in A \subset X$.

Exercise. Prove **Tietze's Extension Theorem**: *Let X be a compact Hausdorff space, $K \subset X$ closed and $f \in C(K)$. Then there exists $F \in C(X)$ such that $F|_K = f$.*

Example. Let (X, τ) be a topological space. Let σ be the $C(X) = C(X, \tau)$-induced topology, i.e. the weakest topology on X making the all τ-continuous functions continuous. Obviously, $\sigma \subset \tau$, and $C(X, \sigma) = C(X, \tau)$. If (X, τ) is a compact Hausdorff space it is easy to check that $\sigma = \tau$.

Example. Let X, Y be topological spaces with bases $\mathcal{B}_X, \mathcal{B}_Y$, respectively. Recall that the product topology for $X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$ has a base

$$\{U \times V \mid U \in \mathcal{B}_X,\ V \in \mathcal{B}_Y\}.$$

This topology is actually induced by the family

$$\{p_X : X \times Y \to X,\ p_Y : X \times Y \to Y\},$$

where the *coordinate projections* p_X and p_Y are defined by $p_X((x, y)) = x$ and $p_Y((x, y)) = y$.

Product topology. Let X_j be a set for every $j \in J$. The *Cartesian product*

$$X = \prod_{j \in J} X_j$$

is the set of the mappings

$$x : J \to \bigcup_{j \in J} X_j \quad \text{such that} \quad \forall j \in J : \; x(j) \in X_j.$$

Due to the Axiom of Choice, X is non-empty if all X_j are non-empty. The mapping

$$p_j : X \to X_j, \quad x \mapsto x_j := x(j),$$

is called the jth *coordinate projection*. Let (X_j, τ_j) be topological spaces. Let $X := \prod_{j \in J} X_j$ be the Cartesian product. Then the $\{p_j \mid j \in J\}$-induced topology on X is called the *product topology* of X.

If $X_j = Y$ for all $j \in J$, it is customary to write

$$\prod_{j \in J} X_j = Y^J = \{f \mid f : J \to Y\}.$$

Weak*-topology. Let $x \mapsto \|x\|$ be the norm of a normed vector space X over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The dual space $X' = \mathcal{L}(X, \mathbb{K})$ of X is set of bounded linear functionals $f : X \to \mathbb{K}$, having a norm

$$\|f\| := \sup_{x \in X: \; \|x\| \leq 1} |f(x)|.$$

This endows X' with a Banach space structure. However, it is often better to use a weaker topology for the dual: Let us define $x(f) := f(x)$ for every $x \in X$ and $f \in X'$; this gives the interpretation $X \subset X'' := \mathcal{L}(X', \mathbb{K})$, because

$$|x(f)| = |f(x)| \leq \|f\| \; \|x\|.$$

So we may treat X as a set of functions $X' \to \mathbb{K}$, and we define the *weak*-topology* of X' to be the X-induced topology of X'.

Let us state a basic property of induced topologies:

Proposition. *Let X have the $\mathcal{F}$-induced topology, and Y be a topological space. A mapping $g : Y \to X$ is continuous if and only if $f \circ g$ is continuous for every $f \in \mathcal{F}$.*

Proof. If g is continuous then the composed mapping $f \circ g$ is continuous for every $f \in \mathcal{F}$.

Conversely, suppose $f_j \circ g$ is continuous for every $f_j \in \mathcal{F}$, $f : X \to X_j$. Let $y \in Y$, $V \subset X$ be open, $g(y) \in V$. Then there exist $\{f_{j_k}\}_{k=1}^n \subset \mathcal{F}$ and open sets $W_{j_k} \subset X_{j_k}$ such that such that

$$g(y) \in \bigcap_{k=1}^{n} f_{j_k}^{-1}(W_{j_k}) \subset V.$$

Let

$$U := \bigcap_{k=1}^{n} (f_{j_k} \circ g)^{-1}(W_{j_k}).$$

Then $U \subset Y$ is open, $y \in U$, and $g(U) \subset V$; hence $g : Y \to X$ is continuous at an arbitrary point $y \in Y$, i.e. $g \in C(Y, X)$ □

Hausdorff preserved in products: It is easy to see that a Cartesian product of Hausdorff spaces is always Hausdorff: If $X = \prod_{j \in J} X_j$ and $x, y \in X$, $x \neq y$, then there exists $j \in J$ such that $x_j \neq y_j$. Therefore there are open sets $U_j, V_j \subset X_j$ such that

$$x_j \in U_j, \quad y_j \in V_j, \quad U_j \cap V_j = \emptyset.$$

Let $U := p_j^{-1}(U_j)$ and $V := p^{-1}(V_j)$. Then $U, V \subset X$ are open,

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

Also compactness is preserved in products; this is stated in Tihonov's Theorem (Tychonoff's Theorem). Before proving this we introduce a tool:

Non-Empty Finite InterSection (NEFIS) property. Let X be a set. Let $NEFIS(X)$ be the set of those families $\mathcal{F} \subset \mathcal{P}(X)$ such that every finite subfamily of $\mathcal{F}$ has a non-empty intersection. In other words, a family $\mathcal{F} \subset \mathcal{P}(X)$ belongs to $NEFIS(X)$ if and only if $\bigcap \mathcal{F}' \neq \emptyset$ for every finite subfamily $\mathcal{F}' \subset \mathcal{F}$.

Lemma. *A topological space X is compact if and only if $\mathcal{F} \notin NEFIS(X)$ whenever $\mathcal{F} \subset \mathcal{P}(X)$ is a family of closed sets satisfying $\bigcap \mathcal{F} = \emptyset$.*

Proof. Let X be a **set**, $\mathcal{U} \subset \mathcal{P}(X)$, and $\mathcal{F} := \{X \setminus U \mid U \in \mathcal{U}\}$. Then

$$\bigcap \mathcal{F} = \bigcap_{U \in \mathcal{U}} (X \setminus U) = X \setminus \bigcup \mathcal{U},$$

so that $\mathcal{U}$ is a cover of X if and only if $\bigcap \mathcal{F} = \emptyset$. Now the claim follows the definition of compactness □

Tihonov's Theorem (1935). *Let X_j be a compact space for every $j \in J$. Then $X = \prod_{j\in J} X_j$ is compact.*

Proof. To avoid the trivial case, suppose $X_j \neq \emptyset$ for every $j \in J$. Let $\mathcal{F} \in NEFIS(X)$ be a family of closed sets. In order to prove the compactness of X we have to show that $\bigcap \mathcal{F} \neq \emptyset$.

Let

$$P := \{\mathcal{G} \in NEFIS(X) \mid \mathcal{F} \subset \mathcal{G}\}.$$

Let us equip the set P with a partial order relation $\leq$:

$$\mathcal{G} \leq \mathcal{H} \overset{\text{definition}}{\Longleftrightarrow} \mathcal{G} \subset \mathcal{H}.$$

The **Hausdorff Maximal Principle** says that the chain $\{\mathcal{F}\} \subset \mathcal{P}$ belongs to a maximal chain $C \subset P$. The reader may verify that $\mathcal{G} := \bigcup C \in P$ is a maximal element of P.

Notice that the maximal element $\mathcal{G}$ may contain non-closed sets. For every $j \in J$ the family

$$\{p_j(G) \mid G \in \mathcal{G}\}$$

belongs to $NEFIS(X_j)$. Define

$$\mathcal{G}_j := \{\overline{p_j(G)} \mid G \in \mathcal{G}\}.$$

Clearly also $\mathcal{G}_j \in NEFIS(X_j)$, and the elements of $\mathcal{G}_j$ are closed sets in X_j. Since X_j is compact, $\bigcap \mathcal{G}_j \neq \emptyset$. Hence we may choose

$$x_j \in \bigcap \mathcal{G}_j.$$

The **Axiom of Choice** provides the existence of the element $x := (x_j)_{j\in J} \in X$. We shall show that $x \in \bigcap \mathcal{F}$, which proves Tihonov's Theorem.

If $V_j \subset X_j$ is a neighborhood of x_j and $G \in \mathcal{G}$ then

$$p_j(G) \cap V_j \neq \emptyset,$$

because $x_j \in \overline{p_j(G)}$. Thus

$$G \cap p_j^{-1}(V_j) \neq \emptyset$$

for every $G \in \mathcal{G}$, so that $\mathcal{G} \cup \{p_j^{-1}(V_j)\}$ belongs to P; the maximality of $\mathcal{G}$ implies that

$$p_j^{-1}(V_j) \in \mathcal{G}.$$

Let $V \in \tau_X$ be a neighborhood of x. Due to the definition of the product topology,

$$x \in \bigcap_{k=1}^{n} p_{j_k}^{-1}(V_{j_k}) \subset V$$

for some finite index set $\{j_k\}_{k=1}^n \subset J$, where $V_{j_k} \subset X_{j_k}$ is a neighborhood of x_{j_k}. Due to the maximality of $\mathcal{G}$, any finite intersection of members of $\mathcal{G}$ belongs to $\mathcal{G}$, so that

$$\bigcap_{k=1}^{n} p_{j_k}^{-1}(V_{j_k}) \in \mathcal{G}.$$

Therefore for every $G \in \mathcal{G}$ and $V \in \mathcal{V}_{\tau_X}(x)$ we have

$$G \cap V \neq \emptyset.$$

Hence $x \in \overline{G}$ for every $G \in \mathcal{G}$, yielding

$$x \in \bigcap_{G \in \mathcal{G}} \overline{G} \overset{\mathcal{F} \subset \mathcal{G}}{\subset} \bigcap_{F \in \mathcal{F}} \overline{F} = \bigcap_{F \in \mathcal{F}} F = \bigcap \mathcal{F},$$

so that $\bigcap \mathcal{F} \neq \emptyset$ □

Remark. Actually, Tihonov's Theorem is equivalent to the Axiom of Choice; we shall not prove this.

Banach–Alaoglu Theorem (1940). *Let X be a normed $\mathbb{C}$-vector space (or a normed $\mathbb{R}$-vector space). The norm-closed unit ball*

$$K := \overline{B_{X'}(0,1)} = \{\phi \in X' : \ \|\phi\|_{X'} \leq 1\}$$

of the dual space X' is weak-compact.*

Proof. Due to Tihonov,

$$P := \prod_{x \in X} \{\lambda \in \mathbb{C} : \ |\lambda| \leq \|x\|\} = \overline{\mathbb{D}(0, \|x\|)}^X$$

is compact in the product topology τ_P. Any element $f \in P$ is a mapping

$$f : X \to \mathbb{C} \quad \text{such that} \quad f(x) \leq \|x\|.$$

Hence $K = X' \cap P$. Let τ_1 and τ_2 be the relative topologies of K inherited from the weak*-topology $\tau_{X'}$ of X' and the product topology τ_P of P, respectively. We shall prove that $\tau_1 = \tau_2$ and that $K \subset P$ is closed; this would show that K is a compact Hausdorff space.

First, let $\phi \in X'$, $f \in P$, $S \subset X$, and $\delta > 0$. Define

$$\begin{aligned} U(\phi, S, \delta) &:= \{\psi \in X' : x \in S \Rightarrow |\psi x - \phi x| < \delta\}, \\ V(f, S, \delta) &:= \{g \in P : x \in S \Rightarrow |g(x) - f(x)| < \delta\}. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{U} &:= \{U(\phi, S, \delta) \mid \phi \in X',\ S \subset X \text{ finite},\ \delta > 0\}, \\ \mathcal{V} &:= \{V(f, S, \delta) \mid f \in P,\ S \subset X \text{ finite},\ \delta > 0\} \end{aligned}$$

are bases for the topologies $\tau_{X'}$ and τ_P, respectively. Clearly

$$K \cap U(\phi, S, \delta) = K \cap V(\phi, S, \delta),$$

so that the topologies $\tau_{X'}$ and τ_P agree on K, i.e. $\tau_1 = \tau_2$.

Still we have to show that $K \subset P$ is closed. Let $f \in \overline{K} \subset P$. First we show that f is linear. Take $x, y \in X$, $\lambda, \mu \in \mathbb{C}$ and $\delta > 0$. Choose $\phi_\delta \in K$ such that

$$f \in V(\phi_\delta, \{x, y, \lambda x + \mu y\}, \delta).$$

Then

$$\begin{aligned} & |f(\lambda x + \mu y) - (\lambda f(x) + \mu f(y))| \\ \leq\ & |f(\lambda x + \mu y) - \phi_\delta(\lambda x + \mu y)| + |\phi_\delta(\lambda x + \mu y) - (\lambda f(x) + \mu f(y))| \\ =\ & |f(\lambda x + \mu y) - \phi_\delta(\lambda x + \mu y)| + |\lambda(\phi_\delta x - f(x)) + \mu(\phi_\delta y - f(y))| \\ \leq\ & |f(\lambda x + \mu y) - \phi_\delta(\lambda x + \mu y)| + |\lambda|\ |\phi_\delta x - f(x)| + |\mu|\ |\phi_\delta y - f(y)| \\ \leq\ & \delta\ (1 + |\lambda| + |\mu|). \end{aligned}$$

This holds for every $\delta > 0$, so that actually

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y),$$

f is linear! Moreover, $\|f\| \leq 1$, because

$$|f(x)| \leq |f(x) - \phi_\delta x| + |\phi_\delta x| \leq \delta + \|x\|.$$

Hence $f \in K$, K is closed □

Remark. The Banach–Alaoglu Theorem implies that a bounded weak*-closed subset of the dual space is a compact Hausdorff space in the relative weak*-topology. However, in a normed space norm-closed balls are compact if and only if the dimension is finite!

Miscellany on (Banach) algebras

Exercise. Let $\{\mathcal{A}_j \mid j \in J\}$ be a family of topological algebras. Endow $\mathcal{A} := \prod_{j\in J} \mathcal{A}_j$ with a structure of a topological algebra.

Before examining commutative Banach algebras in detail, we derive some useful results that could have been treated already in earlier sections (but we forgot to do so :)

Lemma. *Let $\mathcal{A}$ be a commutative algebra and $\mathcal{M}$ be its ideal. Then $\mathcal{M}$ is maximal if and only if $[0]$ is the only non-invertible element of $\mathcal{A}/\mathcal{M}$.*

Proof. Of course, here $[x]$ means $x + \mathcal{M}$, where $x \in \mathcal{A}$. Assume that $\mathcal{M}$ is a maximal ideal. Take $[x] \neq [0]$, so that $x \notin \mathcal{M}$. Define

$$\mathcal{J} := \mathcal{A}x + \mathcal{M} = \{ax + m \mid a \in \mathcal{A},\ m \in \mathcal{M}\}.$$

Then clearly $\mathcal{J} \neq \mathcal{M} \subset \mathcal{J}$, and $\mathcal{J}$ is a vector subspace of $\mathcal{A}$. If $y \in \mathcal{A}$ then

$$\mathcal{J}y = y\mathcal{J} = y\mathcal{A}x + y\mathcal{M} \subset \mathcal{A}x + \mathcal{M} = \mathcal{J},$$

so that either $\mathcal{J}$ is an ideal or $\mathcal{J} = \mathcal{A}$. But since $\mathcal{M}$ is a maximal ideal contained properly in $\mathcal{J}$, we must have $\mathcal{J} = \mathcal{A}$. Thus there exist $a \in \mathcal{A}$ and $m \in \mathcal{M}$ such that $ax + m = \mathbb{I}_{\mathcal{A}}$. Then

$$[a][x] = \mathbb{I}_{\mathcal{A}/\mathcal{M}} = [x][a],$$

$[x]$ is invertible in $\mathcal{A}/\mathcal{M}$.

Conversely, assume that all the non-zero elements are invertible in $\mathcal{A}/\mathcal{M}$. Assume that $\mathcal{J} \subset \mathcal{A}$ is an ideal containing $\mathcal{M}$. **Suppose** $\mathcal{J} \neq \mathcal{M}$, and pick $x \in \mathcal{J} \setminus \mathcal{M}$. Now $[x] \neq [0]$, so that for some $y \in \mathcal{A}$ we have $[x][y] = [\mathbb{I}_{\mathcal{A}}]$. Thereby

$$\mathbb{I}_{\mathcal{A}} \in xy + \mathcal{M} \overset{x\in\mathcal{J}}{\subset} \mathcal{J} + \mathcal{M} \subset \mathcal{J} + \mathcal{J} = \mathcal{J},$$

which is a **contradiction**, since no ideal can contain invertible elements. Therefore we must have $\mathcal{J} = \mathcal{M}$, meaning that $\mathcal{M}$ is maximal □

Proposition. *A maximal ideal in a Banach algebra is closed.*

Proof. In a topological algebra, the closure of an ideal is either an ideal or the whole algebra. Let $\mathcal{M}$ be a maximal ideal of a Banach algebra $\mathcal{A}$. The set $G(\mathcal{A}) \subset \mathcal{A}$ of the invertible elements is open, and $\mathcal{M} \cap G(\mathcal{A}) = \emptyset$ (because no ideal contains invertible elements). Thus $\mathcal{M} \subset \overline{\mathcal{M}} \subset \mathcal{A} \setminus G(\mathcal{A})$, so that $\overline{\mathcal{M}}$ is an ideal containing a maximal ideal $\mathcal{M}$; thus $\overline{\mathcal{M}} = \mathcal{M}$ □

Proposition. *Let $\mathcal{J}$ be a closed ideal of a Banach algebra $\mathcal{A}$. Then the quotient vector space $\mathcal{A}/\mathcal{J}$ is a Banach algebra; moreover, $\mathcal{A}/\mathcal{J}$ is commutative if $\mathcal{A}$ is commutative.*

Proof. Let us denote $[x] := x + \mathcal{J}$ for $x \in \mathcal{A}$. Since $\mathcal{J}$ is a closed vector subspace, the quotient space $\mathcal{A}/\mathcal{J}$ is a Banach space with the norm

$$[x] \mapsto \|[x]\| = \inf_{j \in \mathcal{J}} \|x + j\|.$$

Let $x, y \in \mathcal{A}$ and $\varepsilon > 0$. Then there exist $i, j \in \mathcal{J}$ such that

$$\|x + i\| \leq \|[x]\| + \varepsilon, \quad \|y + j\| \leq \|[y]\| + \varepsilon.$$

Now $(x + i)(y + j) \in [xy]$, so that

$$\begin{aligned} \|[xy]\| &\leq \|(x+i)(y+j)\| \\ &\leq \|x+i\| \, \|y+j\| \\ &\leq (\|[x]\| + \varepsilon) \, (\|[y]\| + \varepsilon) \\ &= \|[x]\| \, \|[y]\| + \varepsilon(\|[x]\| + \|[y]\| + \varepsilon); \end{aligned}$$

since $\varepsilon > 0$ is arbitrary, we have

$$\|[x][y]\| \leq \|[x]\| \, \|[y]\|.$$

Finally, $\|[\mathbb{I}]\| \leq \|\mathbb{I}\| = 1$ and $\|[x]\| = \|[x][\mathbb{I}]\| \leq \|[x]\| \, \|[\mathbb{I}]\|$, so that we have $\|[\mathbb{I}]\| = 1$ □

Exercise*. Let $\mathcal{A}$ be an algebra. The *commutant* of a subset $\mathcal{S} \subset \mathcal{A}$ is

$$\Gamma(\mathcal{S}) := \{x \in \mathcal{A} \mid \forall y \in \mathcal{S} : \ xy = yx\}.$$

Prove the following claims:

(a) $\Gamma(\mathcal{S}) \subset \mathcal{A}$ is a subalgebra; $\Gamma(\mathcal{S})$ is closed if $\mathcal{A}$ is a topological algebra.

(b) $\mathcal{S} \subset \Gamma(\Gamma(\mathcal{S}))$.

(c) If $xy = yx$ for every $x, y \in \mathcal{S}$ then $\Gamma(\Gamma(\mathcal{S})) \subset \mathcal{A}$ is a commutative subalgebra, where $\sigma_{\Gamma(\Gamma(\mathcal{S}))}(z) = \sigma_{\mathcal{A}}(z)$ for every $z \in \Gamma(\Gamma(\mathcal{S}))$.

8 Commutative Banach algebras

In this section we are interested in maximal ideals of commutative Banach algebras. We shall learn that such algebras are closely related to algebras of continuous functions on compact Hausdorff spaces: there is a natural far from trivial homomorphism from a commutative Banach algebra $\mathcal{A}$ to an algebra of functions on the set $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$, which can be endowed with a canonical topology — related mathematics is called the **Gelfand theory**. In the sequel, one should ponder this dilemma: which is more fundamental, a space or algebras of functions on it?

Examples of commutative Banach algebras:

1. Our familiar $C(K)$, when K is a compact space.
2. $L^\infty([0,1])$, when $[0,1]$ is endowed with the Lebesgue measure.
3. $A(\Omega) := C(\overline{\Omega}) \cap H(\Omega)$, when $\Omega \subset \mathbb{C}$ is open and $\overline{\Omega} \subset \mathbb{C}$ is compact.
4. $M(\mathbb{R}^n)$, the convolution algebra of complex Borel measures on $\mathbb{R}^n$, with the Dirac delta distribution at $0 \in \mathbb{R}^n$ as the unit element, and endowed with the total variation norm.
5. The algebra of matrices $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$, where $\alpha, \beta \in \mathbb{C}$; notice that this algebra contains nilpotent elements!

Spectrum of algebra. The *spectrum of an algebra* $\mathcal{A}$ is

$$\mathrm{Spec}(\mathcal{A}) := \mathrm{Hom}(\mathcal{A}, \mathbb{C}),$$

i.e. the set of homomorphisms $\mathcal{A} \to \mathbb{C}$; such a homomorphism is called a *character of* $\mathcal{A}$.

Remark. The concept of spectrum is best suited for **commutative** algebras, as $\mathbb{C}$ is a commutative algebra; here a character $\mathcal{A} \to \mathbb{C}$ should actually be considered as an algebra representation $\mathcal{A} \to \mathcal{L}(\mathbb{C})$. In order to fully capture the structure of a **non-commutative** algebra, we should study representations of type $\mathcal{A} \to \mathcal{L}(X)$, where the vector spaces X are multi-dimensional; for instance, if $\mathcal{H}$ is a Hilbert space of dimension 2 or greater then $\mathrm{Spec}(\mathcal{L}(\mathcal{H})) = \emptyset$. However, the spectrum of a commutative Banach algebra is rich, as there is a bijective correspondence between characters and maximal ideals. Moreover, the spectrum of the algebra is akin to the spectra of its elements:

Theorem (Gelfand, 1940). *Let $\mathcal{A}$ be a commutative Banach algebra. Then:*
(a) *Every maximal ideal of $\mathcal{A}$ is of the form* $\mathrm{Ker}(h)$ *for some* $h \in \mathrm{Spec}(\mathcal{A})$*;*
(b) $\mathrm{Ker}(h)$ *is a maximal ideal for every* $h \in \mathrm{Spec}(\mathcal{A})$*;*
(c) $x \in \mathcal{A}$ *is invertible if and only if* $\forall h \in \mathrm{Spec}(\mathcal{A}): \ h(x) \neq 0$*;*
(d) $x \in \mathcal{A}$ *is invertible if and only if it is not in any ideal of* $\mathcal{A}$*;*
(e) $\sigma(x) = \{h(x) \mid h \in \mathrm{Spec}(\mathcal{A})\}$.

Proof.

(a) Let $\mathcal{M} \subset \mathcal{A}$ be a maximal ideal; let $[x] := x + \mathcal{M}$ for $x \in \mathcal{A}$. Since $\mathcal{A}$ is commutative and $\mathcal{M}$ is maximal, every non-zero element in the quotient algebra $\mathcal{A}/\mathcal{M}$ is invertible. We know that $\mathcal{M}$ is closed, so that $\mathcal{A}/\mathcal{M}$ is a Banach algebra. Due to the Gelfand–Mazur Theorem, there exists an isometric isomorphism $\lambda \in \mathrm{Hom}(\mathcal{A}/\mathcal{M}, \mathbb{C})$. Then

$$h = (x \mapsto \lambda([x])) : \mathcal{A} \to \mathbb{C}$$

is a character, and

$$\mathrm{Ker}(h) = \mathrm{Ker}((x \mapsto [x]) : \mathcal{A} \to \mathcal{A}/\mathcal{M}) = \mathcal{M}.$$

(b) Let $h : \mathcal{A} \to \mathbb{C}$ be a character. Now h is a linear mapping, so that the co-dimension of $\mathrm{Ker}(h)$ in $\mathcal{A}$ equals the dimension of $h(\mathcal{A}) \subset \mathbb{C}$, which clearly is 1. Any ideal of co-dimension 1 in an algebra must be maximal, so that $\mathrm{Ker}(h)$ is maximal.

(c) If $x \in \mathcal{A}$ is invertible and $h \in \mathrm{Spec}(\mathcal{A})$ then $h(x) \in \mathbb{C}$ is invertible, that is $h(x) \neq 0$. For the converse, assume that $x \in \mathcal{A}$ is non-invertible. Then

$$\mathcal{A}x = \{ax \mid a \in \mathcal{A}\}$$

is an ideal of $\mathcal{A}$ (notice that $\mathbb{I} = ax = xa$ would mean that $a = x^{-1}$). Hence by Krull's Theorem, there is a maximal ideal $\mathcal{M} \subset \mathcal{A}$ such that $\mathcal{A}x \subset \mathcal{M}$. Then (a) provides a character $h \in \mathrm{Spec}(\mathcal{A})$ for which $\mathrm{Ker}(h) = \mathcal{M}$. Especially, $h(x) = 0$.

(d) This follows from (a,b,c) directly.

(e) (c) is equivalent to
"$x \in \mathcal{A}$ non-invertible if and only if $\exists h \in \mathrm{Spec}(\mathcal{A}): \ h(x) = 0$",
which is equivalent to
"$\lambda\mathbb{I} - x$ non-invertible if and only if $\exists h \in \mathrm{Spec}(\mathcal{A}): \ h(x) = \lambda$" $\square$

Exercise. Let $\mathcal{A}$ be a Banach algebra and $x, y \in \mathcal{A}$ such that $xy = yx$. Prove that $\sigma(x+y) \subset \sigma(x) + \sigma(y)$ and $\sigma(xy) \subset \sigma(x)\sigma(y)$.

Exercise. Let $\mathcal{A}$ be the algebra of those functions $f : \mathbb{R} \to \mathbb{C}$ for which

$$f(x) = \sum_{n \in \mathbb{Z}} f_n \, e^{ix \cdot n}, \quad \|f\| = \sum_{n \in \mathbb{Z}} |f_n| < \infty.$$

Show that $\mathcal{A}$ is a commutative Banach algebra. Show that if $f \in \mathcal{A}$ and $\forall x \in \mathbb{R} : \; f(x) \neq 0$ then $1/f \in \mathcal{A}$.

Gelfand transform. Let $\mathcal{A}$ be a commutative Banach algebra. The *Gelfand transform* $\widehat{x}$ *of an element* $x \in \mathcal{A}$ is the function

$$\widehat{x} : \mathrm{Spec}(\mathcal{A}) \to \mathbb{C}, \quad \widehat{x}(\phi) := \phi(x).$$

Let $\widehat{\mathcal{A}} := \{\widehat{x} : \mathrm{Spec}(\mathcal{A}) \to \mathbb{C} \mid x \in \mathcal{A}\}$. The mapping

$$\mathcal{A} \to \widehat{\mathcal{A}}, \quad x \mapsto \widehat{x},$$

is called the *Gelfand transform of* $\mathcal{A}$. We endow the set $\mathrm{Spec}(\mathcal{A})$ with the $\widehat{\mathcal{A}}$-induced topology, called the *Gelfand topology*; this topological space is called the *maximal ideal space of* $\mathcal{A}$ (for a good reason, in the light of the previous theorem). In other words, the Gelfand topology is the weakest topology on $\mathrm{Spec}(\mathcal{A})$ making every $\widehat{x}$ a continuous function, i.e. the weakest topology on $\mathrm{Spec}(\mathcal{A})$ for which $\widehat{\mathcal{A}} \subset C(\mathrm{Spec}(\mathcal{A}))$.

Theorem (Gelfand, 1940). *Let $\mathcal{A}$ be a commutative Banach algebra. Then $K = \mathrm{Spec}(\mathcal{A})$ is a compact Hausdorff space in the Gelfand topology, the Gelfand transform is a continuous homomorphism $\mathcal{A} \to C(K)$, and $\|\widehat{x}\| = \sup_{\phi \in K} |\widehat{x}(\phi)| = \rho(x)$ for every $x \in \mathcal{A}$.*

Proof. The Gelfand transform is a homomorphism, since

$$\begin{aligned}
\widehat{\lambda x}(\phi) = \phi(\lambda x) = \lambda \phi(x) = \lambda \widehat{x}(\phi) &= (\lambda \widehat{x})(\phi), \\
\widehat{x+y}(\phi) = \phi(x+y) = \phi(x) + \phi(y) = \widehat{x}(\phi) + \widehat{y}(\phi) &= (\widehat{x} + \widehat{y})(\phi), \\
\widehat{xy}(\phi) = \phi(xy) = \phi(x)\phi(y) = \widehat{x}(\phi)\widehat{y}(\phi) &= (\widehat{x}\widehat{y})(\phi), \\
\widehat{\mathbb{I}_{\mathcal{A}}}(\phi) = \phi(\mathbb{I}_{\mathcal{A}}) = 1 &= \mathbb{I}_{C(K)}(\phi),
\end{aligned}$$

for every $\lambda \in \mathbb{C}$, $x, y \in \mathcal{A}$ and $\phi \in K$. Moreover,

$$\widehat{x}(K) = \{\widehat{x}(\phi) \mid \phi \in K\} = \{\phi(x) \mid \phi \in \mathrm{Spec}(\mathcal{A})\} = \sigma(x),$$

implying

$$\|\widehat{x}\| = \rho(x) \leq \|x\|.$$

Clearly K is a Hausdorff space. What about compactness? Now $K = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$ is a subset of the closed unit ball of the dual Banach space $\mathcal{A}'$; by the Banach–Alaoglu Theorem, this unit ball is compact in the weak*-topology. Recall that the weak*-topology $\tau_{\mathcal{A}'}$ of $\mathcal{A}'$ is the $\mathcal{A}$-induced topology, with the interpretation $\mathcal{A} \subset \mathcal{A}''$; thus the Gelfand topology τ_K is the relative weak*-topology, i.e.

$$\tau_K = \tau_{\mathcal{A}'}|_K.$$

To prove that τ_K is compact, it is sufficient to show that $K \subset \mathcal{A}'$ is closed in the weak*-topology.

Let $f \in \mathcal{A}'$ be in the weak*-closure of K. We have to prove that $f \in K$, i.e.

$$f(xy) = f(x)f(y) \quad \text{and} \quad f(\mathbb{I}) = 1.$$

Let $x, y \in \mathcal{A}$, $\varepsilon > 0$. Let $S := \{\mathbb{I}, x, y, xy\}$. Using the notation of the proof of Banach–Alaoglu Theorem,

$$U(f, S, \varepsilon) = \{\psi \in \mathcal{A}' : \ z \in S \Rightarrow |\psi z - fz| < \varepsilon\}$$

is a weak*-neighborhood of f. Thus choose $h_\varepsilon \in K \cap U(f, S, \varepsilon)$. Then

$$|1 - f(\mathbb{I})| = |h_\varepsilon(\mathbb{I}) - f(\mathbb{I})| < \varepsilon;$$

$\varepsilon > 0$ being arbitrary, we have $f(\mathbb{I}) = 1$. Noticing that $|h_\varepsilon(x)| \leq \|x\|$, we get

$$\begin{aligned}
& |f(xy) - f(x)f(y)| \\
\leq \ & |f(xy) - h_\varepsilon(xy)| + |h_\varepsilon(xy) - h_\varepsilon(x)f(y)| + |h_\varepsilon(x)f(y) - f(x)f(y)| \\
= \ & |f(xy) - h_\varepsilon(xy)| + |h_\varepsilon(x)| \cdot |h_\varepsilon(y) - f(y)| + |h_\varepsilon(x) - f(x)| \cdot |f(y)| \\
\leq \ & \varepsilon \ (1 + \|x\| + |f(y)|).
\end{aligned}$$

This holds for every $\varepsilon > 0$, so that actually

$$f(xy) = f(x)f(y);$$

we have proven that f is a homomorphism, $f \in K$ □

Exercise. Let $\mathcal{A}$ be a commutative Banach algebra. Its *radical* $\mathrm{Rad}(\mathcal{A})$ is the intersection of all the maximal ideals of $\mathcal{A}$. Show that

$$\mathrm{Rad}(\mathcal{A}) = \mathrm{Ker}(x \mapsto \widehat{x}) = \{x \in \ \mathcal{A} \mid \rho(x) = 0\},$$

where $x \mapsto \widehat{x}$ is the Gelfand transform. Show that nilpotent elements of $\mathcal{A}$ belong to the radical.

Exercise. Let X be a finite set. Describe the Gelfand transform of $\mathcal{F}(X)$.

Exercise. Describe the Gelfand transform of the algebra of matrices $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$, where $\alpha, \beta \in \mathbb{C}$.

Theorem. *Let X be a compact Hausdorff space. Then* $\mathrm{Spec}(C(X))$ *is homeomorphic to X.*

Proof. For $x \in X$, let us define the function

$$h_x : C(X) \to \mathbb{C}, \quad f \mapsto f(x) \quad \text{(evaluation at } x \in X\text{)}.$$

This is clearly a homomorphism, and hence we may define the mapping

$$\phi : X \to \mathrm{Spec}(C(X)), \quad x \mapsto h_x.$$

Let us prove that ϕ is a homeomorphism.

If $x, y \in X$, $x \neq y$, then Urysohn's Lemma provides $f \in C(X)$ such that $f(x) \neq f(y)$. Thereby $h_x(f) \neq h_y(f)$, yielding $\phi(x) = h_x \neq h_y = \phi(y)$; thus ϕ is injective. It is also surjective: Namely, let us **assume** that $h \in \mathrm{Spec}(C(X)) \setminus \phi(X)$. Now $\mathrm{Ker}(h) \subset C(X)$ is a maximal ideal, and for every $x \in X$ we may choose

$$f_x \in \mathrm{Ker}(h) \setminus \mathrm{Ker}(h_x) \subset C(X).$$

Then $U_x := f_x^{-1}(\mathbb{C} \setminus \{0\}) \in \mathcal{V}(x)$, so that

$$\mathcal{U} = \{U_x \mid x \in X\}$$

is an open cover of X, which due to the compactness has a finite subcover $\{U_{x_j}\}_{j=1}^n \subset \mathcal{U}$. Since $f_{x_j} \in \mathrm{Ker}(h)$, the function

$$f := \sum_{j=1}^n |f_{x_j}|^2 = \sum_{j=1}^n f_{x_j} \overline{f_{x_j}}$$

belongs to $\mathrm{Ker}(h)$. Clearly $f(x) \neq 0$ for every $x \in X$. Therefore $g \in C(X)$ with $g(x) = 1/f(x)$ is the inverse element of f; this is a **contradiction**, since no invertible element belongs to an ideal. Thus ϕ must be surjective.

We have proven that $\phi : X \to \mathrm{Spec}(C(X))$ is a bijection. Thereby X and $\mathrm{Spec}(C(X))$ can be identified as sets. The Gelfand-topology of $\mathrm{Spec}(C(X))$ is then identified with the $C(X)$-induced topology σ of X, which is weaker than the original topology τ of X. Hence $\phi : (X, \tau) \to \mathrm{Spec}(C(X))$ is continuous. Actually, $\sigma = \tau$, because a continuous bijection from a compact space to a Hausdorff space is a homeomorphism $\square$

Corollary. *Let X and Y be compact Hausdorff spaces. Then the Banach algebras $C(X)$ and $C(Y)$ are isomorphic if and only if X is homeomorphic to Y.*

Proof. By the previous Theorem, $X \cong \mathrm{Spec}(C(X))$ and $Y \cong \mathrm{Spec}(C(Y))$. If $C(X)$ and $C(Y)$ are isomorphic Banach algebras then

$$X \cong \mathrm{Spec}(C(X)) \stackrel{C(X)\cong C(Y)}{\cong} \mathrm{Spec}(C(Y)) \cong Y.$$

Conversely, a homeomorphism $\phi : X \to Y$ begets a Banach algebra isomorphism

$$\Phi : C(Y) \to C(X), \quad (\Phi f)(x) := f(\phi(x)),$$

as the reader easily verifies □

9 Polynomial approximations: Stone–Weierstrass

In this section we study densities of subalgebras in $C(X)$. These results will be applied in characterizing function algebras among Banach algebras. First we study continuous functions on $[a,b] \subset \mathbb{R}$:

Weierstrass Theorem (1885). *Polynomials are dense in* $C([a,b])$.

Proof. Evidently, it is enough to consider the case $[a,b]=[0,1]$. Let $f \in C([0,1])$, and let $g(x) = f(x) - (f(0) + (f(1)-f(0))x)$; then $g \in C(\mathbb{R})$ if we define $g(x)=0$ for $x \in \mathbb{R} \setminus [0,1]$. For $n \in \mathbb{N}$ let us define $k_n : \mathbb{R} \to [0,\infty[$ by

$$k_n(x) := \begin{cases} \frac{(1-x^2)^n}{\int_{-1}^1 (1-t^2)^n \ \mathrm{d}t}, & \text{when } |x|<1, \\ 0, & \text{when } |x| \geq 1. \end{cases}$$

Then define $P_n := g * k_n$ (*convolution* of g and k_n), that is

$$\begin{aligned} P_n(x) &= \int_{-\infty}^{\infty} g(x-t)\ k_n(t)\ \mathrm{d}t = \int_{-\infty}^{\infty} g(t)\ k_n(x-t)\ \mathrm{d}t \\ &= \int_0^1 g(t)\ k_n(x-t)\ \mathrm{d}t, \end{aligned}$$

and from this last formula we see that P_n is a polynomial on $[0,1]$. Notice that P_n is real-valued if f is real-valued. Take any $\varepsilon > 0$. Function g is uniformly continuous, so that there exists $\delta > 0$ such that

$$\forall x,y \in \mathbb{R}: \ |x-y|<\delta \Rightarrow |g(x)-g(y)|<\varepsilon.$$

Let $\|g\| = \max_{t \in [0,1]} |g(t)|$. Take $x \in [0,1]$. Then

$$\begin{aligned} |P_n(x)-g(x)| &= \left| \int_{-\infty}^{\infty} g(x-t)\ k_n(t)\ \mathrm{d}t - g(x) \int_{-\infty}^{\infty} k_n(t)\ \mathrm{d}t \right| \\ &= \left| \int_{-1}^{1} (g(x-t)-g(x))\ k_n(t)\ \mathrm{d}t \right| \\ &\leq \int_{-1}^{1} |g(x-t)-g(x)|\ k_n(t)\ \mathrm{d}t \\ &\leq \int_{-1}^{-\delta} 2\|g\|\ k_n(t)\ \mathrm{d}t + \int_{-\delta}^{\delta} \varepsilon\ k_n(t)\ \mathrm{d}t + \int_{\delta}^{1} 2\|g\|\ k_n(t)\ \mathrm{d}t \\ &\leq 4\|g\| \int_{\delta}^{1} k_n(t)\ \mathrm{d}t + \varepsilon. \end{aligned}$$

The reader may verify that $\int_\delta^1 k_n(t)\ \mathrm{d}t \ \to_{n\to\infty} 0$ for every $\delta > 0$. Hence $\|Q_n - f\| \to_{n\to\infty} 0$, where $Q_n(x) = P_n(x) + f(0) + (f(1)-f(0))x$ $\square$

Exercise. Show that the last claim in the proof of Weierstrass Theorem is true.

For $f : X \to \mathbb{C}$ let us define $f^* : X \to \mathbb{C}$ by $f^*(x) := \overline{f(x)}$, and define $|f| : X \to \mathbb{C}$ by $|f|(x) := |f(x)|$. A subalgebra $\mathcal{A} \subset \mathcal{F}(X)$ is called *involutive* if $f^* \in \mathcal{A}$ whenever $f \in \mathcal{A}$.

Stone–Weierstrass Theorem (1937). *Let X be a compact space. Let $\mathcal{A} \subset C(X)$ be an involutive subalgebra separating the points of X. Then $\mathcal{A}$ is dense in $C(X)$.*

Proof. If $f \in \mathcal{A}$ then $f^* \in \mathcal{A}$, so that the real part $\Re f = \dfrac{f + f^*}{2}$ belongs to $\mathcal{A}$. Let us define

$$\mathcal{A}_\mathbb{R} := \{\Re f \mid f \in \mathcal{A}\};$$

this is a $\mathbb{R}$-subalgebra of the $\mathbb{R}$-algebra $C(X, \mathbb{R})$ of continuous real-valued functions on X. Then

$$\mathcal{A} = \{f + ig \mid f, g \in \mathcal{A}_\mathbb{R}\},$$

so that $\mathcal{A}_\mathbb{R}$ separates the points of X. If we can show that $\mathcal{A}_\mathbb{R}$ is dense in $C(X, \mathbb{R})$ then $\mathcal{A}$ would be dense in $C(X)$.

First we have to show that $\overline{\mathcal{A}_\mathbb{R}}$ is closed under taking maximums and minimums. For $f, g \in C(X, \mathbb{R})$ we define

$$\max(f, g)(x) := \max(f(x), g(x)), \quad \min(f, g)(x) := \min(f(x), g(x)).$$

Notice that $\overline{\mathcal{A}_\mathbb{R}}$ is an algebra over the field $\mathbb{R}$. Since

$$\max(f, g) = \frac{f + g}{2} + \frac{|f - g|}{2}, \quad \min(f, g) = \frac{f + g}{2} - \frac{|f - g|}{2},$$

it is enough to prove that $|h| \in \overline{\mathcal{A}_\mathbb{R}}$ whenever $h \in \overline{\mathcal{A}_\mathbb{R}}$. Let $h \in \overline{\mathcal{A}_\mathbb{R}}$. By the Weierstrass Theorem there is a sequence of polynomials $P_n : \mathbb{R} \to \mathbb{R}$ such that

$$P_n(x) \to_{n \to \infty} |x|$$

uniformly on the interval $[-\|h\|, \|h\|]$. Thereby

$$\||h| - P_n(h)\| \to_{n \to \infty} 0,$$

where $P_n(h)(x) := P_n(h(x))$. Since $P_n(h) \in \overline{\mathcal{A}_\mathbb{R}}$ for every n, this implies that $|h| \in \overline{\mathcal{A}_\mathbb{R}}$. Now we know that $\max(f, g), \min(f, g) \in \overline{\mathcal{A}_\mathbb{R}}$ whenever $f, g \in \overline{\mathcal{A}_\mathbb{R}}$.

Now we are ready to prove that $f \in C(X, \mathbb{R})$ can be approximated by elements of $\mathcal{A}_{\mathbb{R}}$. Take $\varepsilon > 0$ and $x, y \in X$, $x \neq y$. Since $\mathcal{A}_{\mathbb{R}}$ separates the points of X, we may pick $h \in \mathcal{A}_{\mathbb{R}}$ such that $h(x) \neq h(y)$. Let $g_{xx} = f(x)\mathbb{I}$, and let

$$g_{xy}(z) := \frac{h(z) - h(y)}{h(x) - h(y)}\, f(x) + \frac{h(z) - h(x)}{h(y) - h(x)}\, f(y).$$

Here $g_{xx}, g_{xy} \in \mathcal{A}_{\mathbb{R}}$, since $\mathcal{A}_{\mathbb{R}}$ is an algebra. Furthermore,

$$g_{xy}(x) = f(x), \quad g_{xy}(y) = f(y).$$

Due to the continuity of g_{xy}, there is an open set $V_{xy} \in \mathcal{V}(y)$ such that

$$z \in V_{xy} \;\Rightarrow\; f(z) - \varepsilon < g_{xy}(z).$$

Now $\{V_{xy} \mid y \in X\}$ is an open cover of the compact space X, so that there is a finite subcover $\{V_{xy_j} \mid 1 \leq j \leq n\}$. Define

$$g_x := \max_{1 \leq j \leq n} g_{xy_j};$$

$g_x \in \overline{\mathcal{A}_{\mathbb{R}}}$, because $\overline{\mathcal{A}_{\mathbb{R}}}$ is closed under taking maximums. Moreover,

$$\forall z \in X: \; f(z) - \varepsilon < g_x(z).$$

Due to the continuity of g_x (and since $g_x(x) = f(x)$), there is an open set $U_x \in \mathcal{V}(x)$ such that

$$z \in U_x \;\Rightarrow\; g_x(z) < f(z) + \varepsilon.$$

Now $\{U_x \mid x \in X\}$ is an open cover of the compact space X, so that there is a finite subcover $\{U_{x_i} \mid 1 \leq i \leq m\}$. Define

$$g := \min_{1 \leq i \leq m} g_{x_i};$$

$g \in \overline{\mathcal{A}_{\mathbb{R}}}$, because $\overline{\mathcal{A}_{\mathbb{R}}}$ is closed under taking minimums. Moreover,

$$\forall z \in X: \; g(z) < f(z) + \varepsilon.$$

Thus

$$f(z) - \varepsilon < \min_{1 \leq i \leq m} g_{x_i}(z) = g(z) < f(z) + \varepsilon,$$

that is $|g(z) - f(z)| < \varepsilon$ for every $z \in X$, i.e. $\|g - f\| < \varepsilon$. Hence $\mathcal{A}_{\mathbb{R}}$ is dense in $C(X, \mathbb{R})$ implying that $\mathcal{A}$ is dense in $C(X)$ □

Remark. Notice that under the assumptions of the Stone–Weierstrass Theorem, the compact space is actually a compact Hausdorff space, since continuous functions separate the points.

Exercise*. Let K be a compact Hausdorff space, $\emptyset \neq S \subset K$, and $\mathcal{J} \subset C(K)$ be an ideal. Let us define

$$\mathcal{I}(S) := \{f \in C(K) \mid \forall x \in S: \ f(x) = 0\},$$
$$V(\mathcal{J}) := \{x \in K \mid \forall f \in \mathcal{J}: \ f(x) = 0\}.$$

Prove that
(a) $\mathcal{I}(S) \subset C(K)$ a closed ideal,
(b) $V(\mathcal{J}) \subset K$ is a closed non-empty subset,
(c) $V(\mathcal{I}(S)) = \overline{S}$ (hint: Urysohn), and
(d) $\mathcal{I}(V(\mathcal{J})) = \overline{\mathcal{J}}$.
Lesson to be learned:
topology of K goes hand in hand with the (closed) ideal structure of $C(K)$.

10 C*-algebras

Now we are finally in the position to abstractly characterize algebras $C(X)$ among Banach algebras: according to Gelfand and Naimark, the category of compact Hausdorff spaces is equivalent to the category of commutative C*-algebras. The class of C*-algebras behaves nicely, and the related functional analysis adequately deserves the name "non-commutative topology".

Involutive algebra. An algebra $\mathcal{A}$ is a $*$-*algebra* ("*star-algebra*" or an *involutive algebra*) if there is a mapping $(x \mapsto x^*) : \mathcal{A} \to \mathcal{A}$ satisfying

$$(\lambda x)^* = \overline{\lambda} x^*, \quad (x+y)^* = x^* + y^*, \quad (xy)^* = y^* x^*, \quad (x^*)^* = x$$

for every $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$; such a mapping is called an *involution*. In other words, an involution is a conjugate-linear anti-multiplicative self-invertible mapping $\mathcal{A} \to \mathcal{A}$.

A $*$-*homomorphism* $\phi : \mathcal{A} \to \mathcal{B}$ between involutive algebras $\mathcal{A}$ and $\mathcal{B}$ is an algebra homomorphism satisfying

$$\phi(x^*) = \phi(x)^*$$

for every $x \in \mathcal{A}$. The set of all $*$-homomorphisms between $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathrm{Hom}^*(\mathcal{A}, \mathcal{B})$.

C*-algebra. A *C*-algebra* $\mathcal{A}$ is an involutive Banach algebra such that

$$\|x^* x\| = \|x\|^2$$

for every $x \in \mathcal{A}$.

Examples.

1. The Banach algebra $\mathbb{C}$ is a C*-algebra with the involution $\lambda \mapsto \lambda^* = \overline{\lambda}$, i.e. the complex conjugation.

2. If K is a compact space then $C(K)$ is a commutative C*-algebra with the involution $f \mapsto f^*$ by complex conjugation, $f^*(x) := \overline{f(x)}$.

3. $L^\infty([0,1])$ is a C*-algebra, when the involution is as above.

4. $A(\mathbb{D}(0,1)) = C\left(\overline{\mathbb{D}(0,1)}\right) \cap H(\mathbb{D}(0,1))$ is an involutive Banach algebra with $f^*(z) := \overline{f(\overline{z})}$, but it is not a C*-algebra.

5. The radical of a commutative C*-algebra is always the trivial $\{0\}$, and thus 0 is the only nilpotent element. Hence for instance the algebra of matrices $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ (where $\alpha, \beta \in \mathbb{C}$) cannot be a C*-algebra.

6. If $\mathcal{H}$ is a Hilbert space then $\mathcal{L}(\mathcal{H})$ is a C*-algebra when the involution is the usual adjunction $A \mapsto A^*$, and clearly any norm-closed involutive subalgebra of $\mathcal{L}(\mathcal{H})$ is also a C*-algebra. Actually, there are no others, but we shall not prove this fact in these lecture notes:

Gelfand–Naimark Theorem (1943). *If $\mathcal{A}$ is a C*-algebra then there exists a Hilbert space $\mathcal{H}$ and an isometric $*$-homomorphism onto a closed involutive subalgebra of $\mathcal{L}(\mathcal{H})$* □

However, we shall characterize the commutative case: the Gelfand transform of a commutative C*-algebra $\mathcal{A}$ will turn out to be an isometric isomorphism $\mathcal{A} \to C(\mathrm{Spec}(\mathcal{A}))$, so that $\mathcal{A}$ "is" the function algebra $C(K)$ for the compact Hausdorff space $K = \mathrm{Spec}(\mathcal{A})$! Before going into this, we prove some related results.

Proposition. *Let $\mathcal{A}$ be a $*$-algebra. Then $\mathbb{I}^* = \mathbb{I}$, $x \in \mathcal{A}$ is invertible if and only if $x^* \in \mathcal{A}$ is invertible, and $\sigma(x^*) = \overline{\sigma(x)} := \{\overline{\lambda} \mid \lambda \in \sigma(x)\}$.*

Proof. First,

$$\mathbb{I}^* = \mathbb{I}^*\mathbb{I} = \mathbb{I}^*(\mathbb{I}^*)^* = (\mathbb{I}^*\mathbb{I})^* = (\mathbb{I}^*)^* = \mathbb{I};$$

second,

$$(x^{-1})^*x^* = (xx^{-1})^* = \mathbb{I}^* = \mathbb{I} = \mathbb{I}^* = (x^{-1}x)^* = x^*(x^{-1})^*;$$

third,

$$\overline{\lambda}\mathbb{I} - x^* = (\lambda\mathbb{I}^*)^* - x^* = (\lambda\mathbb{I})^* - x^* = (\lambda\mathbb{I} - x)^*,$$

which concludes the proof □

Proposition. *Let $\mathcal{A}$ be a C*-algebra, and $x = x^* \in \mathcal{A}$. Then $\sigma(x) \subset \mathbb{R}$.*

Proof. **Assume** that $\lambda \in \sigma(x) \setminus \mathbb{R}$, i.e. $\lambda = \lambda_1 + i\lambda_2$ for some $\lambda_j \in \mathbb{R}$ with $\lambda_2 \neq 0$. Hence we may define $y := (x - \lambda_1\mathbb{I})/\lambda_2 \in \mathcal{A}$. Now $y^* = y$. Moreover, $i \in \sigma(y)$, because

$$i\mathbb{I} - y = \frac{\lambda\mathbb{I} - x}{\lambda_2}.$$

Take $t \in \mathbb{R}$. Then $t+1 \in \sigma(t\mathbb{I} - iy)$, because

$$(t+1)\mathbb{I} - (t\mathbb{I} - iy) = -i(i\mathbb{I} - y).$$

Thereby

$$\begin{aligned}
(t+1)^2 &\leq \rho(t\mathbb{I} - iy)^2 \\
&\leq \|t\mathbb{I} - iy\|^2 \\
&\overset{C^*}{=} \|(t\mathbb{I} - iy)^*(t\mathbb{I} - iy)\| \\
&\overset{t\in\mathbb{R},\ y^*=y}{=} \|(t\mathbb{I} + iy)(t\mathbb{I} - iy)\| \ = \ \|t^2\mathbb{I} + y^2\| \\
&\leq t^2 + \|y\|,
\end{aligned}$$

so that $2t+1 \leq \|y\|$ for every $t \in \mathbb{R}$; a **contradiction** □

Corollary. *Let $\mathcal{A}$ a C^*-algebra, $\phi : \mathcal{A} \to \mathbb{C}$ a homomorphism, and $x \in \mathcal{A}$. Then $\phi(x^*) = \overline{\phi(x)}$, i.e. ϕ is a $*$-homomorphism.*

Proof. Define the "real part" and the "imaginary part" of x by

$$u := \frac{x + x^*}{2}, \quad v := \frac{x - x^*}{2i}.$$

Then $x = u + iv$, $u^* = u$, $v^* = v$, and $x^* = u - iv$. Since a homomorphism maps invertibles to invertibles, we have $\phi(u) \in \sigma(u)$; we know that $\sigma(u) \subset \mathbb{R}$, because $u^* = u$. Similarly we obtain $\phi(v) \in \mathbb{R}$. Thereby

$$\phi(x^*) = \phi(u - iv) = \phi(u) - i\phi(v) = \overline{\phi(u) + i\phi(v)} = \overline{\phi(u + iv)} = \overline{\phi(x)};$$

this means that $\mathrm{Hom}^*(\mathcal{A}, \mathbb{C}) = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$ □

Exercise. Let $\mathcal{A}$ be a Banach algebra, $\mathcal{B}$ its closed subalgebra, and $x \in \mathcal{B}$. Prove the following facts:
(a) $G(\mathcal{B})$ is open and closed in $G(\mathcal{A}) \cap \mathcal{B}$.
(b) $\sigma_{\mathcal{A}}(x) \subset \sigma_{\mathcal{B}}(x)$ and $\partial\sigma_{\mathcal{B}}(x) \subset \partial\sigma_{\mathcal{A}}(x)$.
(c) If $\mathbb{C} \setminus \sigma_{\mathcal{A}}(x)$ is connected then $\sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{B}}(x)$.

Using the results of the exercise above, the reader can prove the following important fact on the invariance of spectrum in C*-algebras:

Exercise*. Let $\mathcal{A}$ be a C*-algebra and $\mathcal{B}$ its C*-subalgebra. Show that $\sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{A}}(x)$ for every $x \in \mathcal{B}$.

Lemma. *Let $\mathcal{A}$ be a C*-algebra. Then $\|x\|^2 = \rho(x^*x)$ for every $x \in \mathcal{A}$.*

Proof. Now

$$\|(x^*x)^2\| = \|(x^*x)(x^*x)\| = \|(x^*x)^*(x^*x)\| \overset{C^*}{=} \|x^*x\|^2,$$

so that by induction

$$\|(x^*x)^{2^n}\| = \|x^*x\|^{2^n}$$

for every $n \in \mathbb{N}$. Therefore applying the Spectral Radius Formula, we get

$$\rho(x^*x) = \lim_{n\to\infty} \|(x^*x)^{2^n}\|^{1/2^n} = \lim_{n\to\infty} \|x^*x\|^{2^n/2^n} = \|x^*x\|,$$

the result we wanted □

Exercise*. Let $\mathcal{A}$ be a C*-algebra. Show that there can be at most one C*-algebra norm on an involutive Banach algebra. Moreover, prove that if $\mathcal{A}, \mathcal{B}$ are C*-algebras then $\phi \in \mathrm{Hom}^*(\mathcal{A}, \mathcal{B})$ is continuous and has a norm $\|\phi\| = 1$.

Commutative Gelfand–Naimark. *Let $\mathcal{A}$ be a commutative C*-algebra. Then the Gelfand transform $(x \mapsto \widehat{x}) : \mathcal{A} \to C(\mathrm{Spec}(\mathcal{A}))$ is an isometric $*$-isomorphism.*

Proof. Let $K = \mathrm{Spec}(\mathcal{A})$. We already know that the Gelfand transform is a Banach algebra homomorphism $\mathcal{A} \to C(K)$. Let $x \in \mathcal{A}$ and $\phi \in K$. Since ϕ is actually a $*$-homomorphism, we get

$$\widehat{x^*}(\phi) = \phi(x^*) = \overline{\phi(x)} = \overline{\widehat{x}(\phi)} = \widehat{x}^*(\phi);$$

the Gelfand transform is a $*$-homomorphism.

Now we have proven that $\widehat{\mathcal{A}} \subset C(K)$ is an involutive subalgebra separating the points of K. Stone–Weierstrass Theorem thus says that $\widehat{\mathcal{A}}$ is dense in $C(K)$. If we can show that the Gelfand transform $\mathcal{A} \to \widehat{\mathcal{A}}$ is an isometry then we must have $\widehat{\mathcal{A}} = C(K)$: Take $x \in \mathcal{A}$. Then

$$\|\widehat{x}\|^2 = \|\widehat{x}^*\widehat{x}\| = \|\widehat{x^*x}\| \overset{\text{Gelfand}}{=} \rho(x^*x) \overset{\text{Lemma}}{=} \|x\|^2,$$

i.e. $\|\widehat{x}\| = \|x\|$ □

Exercise*. Show that an injective $*$-homomorphism between C*-algebras is an isometry. (Hint: Gelfand transform.)

Exercise*. A linear functional f on a C*-algebra $\mathcal{A}$ is called *positive* if $f(x^*x) \geq 0$ for every $x \in \mathcal{A}$. Show that the positive functionals separate the points of $\mathcal{A}$.

Exercise*. Prove that the involution of a C*-algebra cannot be altered without destroying the C*-property $\|x^*x\| = \|x\|^2$.

An element x of a C*-algebra is called *normal* if $x^*x = xx^*$. We use the commutative Gelfand–Naimark Theorem to create the so called continuous functional calculus at a normal element — a non-commutative C*-algebra admits some commutative studies:

Theorem. *Let $\mathcal{A}$ be a C*-algebra, and $x \in \mathcal{A}$ be a normal element. Let $\iota = (\lambda \mapsto \lambda) : \sigma(x) \to \mathbb{C}$. Then there exists a unique isometric $*$-homomorphism $\phi : C(\sigma(x)) \to \mathcal{A}$ such that $\phi(\iota) = x$ and $\phi(C(\sigma(x)))$ is the C*-algebra generated by x, i.e. the smallest C*-algebra containing $\{x\}$.*

Proof. Let $\mathcal{B}$ be the C*-algebra generated by x. Since x is normal, $\mathcal{B}$ is commutative. Let $\mathrm{Gel} = (y \mapsto \widehat{y}) : \mathcal{B} \to C(\mathrm{Spec}(\mathcal{B}))$ be the Gelfand transform of $\mathcal{B}$. The reader may easily verify that

$$\widehat{x} : \mathrm{Spec}(\mathcal{B}) \to \sigma(x)$$

is a continuous bijection from a compact space to a Hausdorff space; hence it is a homeomorphism. Let us define the mapping

$$C_{\widehat{x}} : C(\sigma(x)) \to C(\mathrm{Spec}(\mathcal{B})), \quad (C_{\widehat{x}}f)(h) := f(\widehat{x}(h)) = f(h(x));$$

$C_{\widehat{x}}$ can be thought as a “transpose” of $\widehat{x}$. Let us define

$$\phi = \mathrm{Gel}^{-1} \circ C_{\widehat{x}} : C(\sigma(x)) \to \mathcal{B} \subset \mathcal{A}.$$

Then $\phi : C(\sigma(x) \to \mathcal{A}$ is obviously an isometric $*$-homomorphism. Furthermore,

$$\phi(\iota) = \mathrm{Gel}^{-1}(C_{\widehat{x}}(\iota)) = \mathrm{Gel}^{-1}(\widehat{x}) = \mathrm{Gel}^{-1}(\mathrm{Gel}(x)) = x.$$

Due to the Stone–Weierstrass Theorem, the $*$-algebra generated by $\iota \in C(\sigma(x))$ is dense in $C(\sigma(x))$; since the $*$-homomorphism ϕ maps the generator ι to the generator x, the uniqueness of ϕ follows $\square$

Remark. The $*$-homomorphism $\phi : C(\sigma(x)) \to \mathcal{A}$ in above is called the *(continuous) functional calculus at the normal element* $\phi(\iota) = x \in \mathcal{A}$. If $p = (z \mapsto \sum_{j=1}^{n} a_j z^j) : \mathbb{C} \to \mathbb{C}$ is a polynomial then it is natural to define $p(x) := \sum_{j=1}^{n} a_j x^j$. Then actually

$$p(x) = \phi(p);$$

hence it is natural to define $f(x) := \phi(f)$ for every $f \in C(\sigma(x))$. It is easy to check that if $f \in C(\sigma(x))$ and $h \in \mathrm{Spec}(\mathcal{B})$ then $f(h(x)) = h(f(x))$.

Exercise. Let $\mathcal{A}$ be a C*-algebra, $x \in \mathcal{A}$ normal, $f \in C(\sigma(x))$, and $g \in C(f(\sigma(x)))$. Show that $\sigma(f(x)) = f(\sigma(x))$ and that $(g \circ f)(x) = g(f(x))$.

11 Metrizability

Next we try to construct metrics on compact spaces. We shall learn that a compact space is metrizable if and only if the corresponding commutative C*-algebra is separable. Metrizability is equivalent to the existence of a countable family of continuous functions separating the points of the space. As a vague analogy to the manifolds, the reader may view such a countable family as a set of coordinate functions on the space.

Theorem. *If $\mathcal{F} \subset C(X)$ is a countable family separating the points of a compact space (X, τ) then X is metrizable.*

Proof. Let $\mathcal{F} = \{f_n\}_{n=0}^{\infty} \subset C(X)$ separate the points of X. We can assume that $\|f_n\| \leq 1$ for every $n \in \mathbb{N}$; otherwise consider for instance functions $x \mapsto f_n(x)/(1 + |f_n(x)|)$. Let us define

$$d(x, y) := \sup_{n \in \mathbb{N}} 2^{-n} |f_n(x) - f_n(y)|$$

for every $x, y \in X$. Next we prove that $d : X \times X \to [0, \infty[$ is a metric: $d(x, y) = 0 \Leftrightarrow x = y$, because $\{f_n\}_{n=0}^{\infty}$ is a separating family. Clearly also $d(x, y) = d(y, x)$ for every $x, y \in X$. Let $x, y, z \in X$. We have the triangle inequality:

$$\begin{aligned} d(x, z) &= \sup_{n \in \mathbb{N}} 2^{-n} |f_n(x) - f_n(z)| \\ &\leq \sup_{n \in \mathbb{N}} (2^{-n} |f_n(x) - f_n(y)| + 2^{-n} |f_n(y) - f_n(z)|) \\ &\leq \sup_{m \in \mathbb{N}} 2^{-m} |f_m(x) - f_m(y)| + \sup_{n \in \mathbb{N}} 2^{-n} |f_n(y) - f_n(z)| \\ &= d(x, y) + d(y, z). \end{aligned}$$

Hence d is a metric on X.

Finally, let us prove that the metric topology coincides with the original topology, $\tau_d = \tau$: Let $x \in X$, $\varepsilon > 0$. Take $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon$. Define

$$U_n := f_n^{-1}(\mathbb{D}(f_n(x), \varepsilon)) \in \mathcal{V}_\tau(x), \quad U := \bigcap_{n=0}^{N} U_n \in \mathcal{V}_\tau(x).$$

If $y \in U$ then

$$d(x, y) = \sup_{n \in \mathbb{N}} 2^{-n} |f_n(x) - f_n(y)| < \varepsilon.$$

Thus $x \in U \subset B_d(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$. This proves that the original topology τ is finer than the metric topology τ_d, i.e. $\tau_d \subset \tau$. Combined with the facts that (X, τ) is compact and (X, τ_d) is Hausdorff, this implies that we must have $\tau_d = \tau$ $\square$

Corollary. *Let X be a compact Hausdorff space. Then X is metrizable if and only if it has a countable basis.*

Proof. Suppose X is a compact space, metrizable with a metric d. Let $r > 0$. Then $\mathcal{B}_r = \{B_d(x,r) \mid x \in X\}$ is an open cover of X, thus having a finite subcover $\mathcal{B}'_r \subset \mathcal{B}_r$. Then $\mathcal{B} := \bigcup_{n=1}^{\infty} \mathcal{B}'_{1/n}$ is a countable basis for X.

Conversely, suppose X is a compact Hausdorff space with a countable basis $\mathcal{B}$. Then the family

$$\mathcal{C} := \{(B_1, B_2) \in \mathcal{B} \times \mathcal{B} \mid \overline{B_1} \subset B_2\}$$

is countable. For each $(B_1, B_2) \in \mathcal{C}$ Urysohn's Lemma provides a function $f_{B_1 B_2} \in C(X)$ satisfying

$$f_{B_1 B_2}(\overline{B_1}) = \{0\} \quad \text{and} \quad f_{B_1 B_2}(X \setminus B_2) = \{1\}.$$

Next we show that the countable family

$$\mathcal{F} = \{f_{B_1 B_2} : \ (B_1, B_2) \in \mathcal{C}\} \subset C(X)$$

separates the points of X: Take $x, y \in X$, $x \neq y$. Then $W := X \setminus \{y\} \in \mathcal{V}(x)$. Since X is a compact Hausdorff space, there exists $U \in \mathcal{V}(x)$ such that $\overline{U} \subset W$. Take $B', B \in \mathcal{B}$ such that $x \in B' \subset \overline{B'} \subset B \subset U$. Then $f_{B'B}(x) = 0 \neq 1 = f_{B'B}(y)$. Thus X is metrizable □

Conclusion. *Let X be a compact Hausdorff space. Then X is metrizable if and only if $C(X)$ is separable (i.e. contains a countable dense subset).*

Proof. Suppose X is a metrizable compact space. Let $\mathcal{F} \subset C(X)$ be a countable family separating the points of X (as in the proof of the previous Corollary). Let $\mathcal{G}$ be the set of finite products of functions f for which $f \in \mathcal{F} \cup \mathcal{F}^* \cup \{\mathbb{I}\}$; the set $\mathcal{G} = \{g_j\}_{j=0}^{\infty}$ is countable. The linear span $\mathcal{A}$ of $\mathcal{G}$ is the involutive algebra generated by $\mathcal{F}$ (the smallest $*$-algebra containing $\mathcal{F}$); due to the Stone–Weierstrass Theorem, $\mathcal{A}$ is dense in $C(X)$. If $S \subset \mathbb{C}$ is a countable dense set then

$$\{\lambda_0 \mathbb{I} + \sum_{j=1}^{n} \lambda_j g_j \mid n \in \mathbb{Z}^+, \ (\lambda_j)_{j=0}^{n} \subset S\}$$

is a countable dense subset of $\mathcal{A}$, thereby dense in C(X).

Conversely, assume that $\mathcal{F} = \{f_n\}_{n=0}^{\infty} \subset C(X)$ is a dense subset. Take $x, y \in X$, $x \neq y$. By Urysohn's Lemma there exists $f \in C(X)$ such that $f(x) = 0 \neq 1 = f(y)$. Take $f_n \in \mathcal{F}$ such that $\|f - f_n\| < 1/2$. Then

$$|f_n(x)| < 1/2 \quad \text{and} \quad |f_n(y)| > 1/2,$$

so that $f_n(x) \neq f_n(y)$; $\mathcal{F}$ separates the points of X □

Exercise*. Prove that a topological space with a countable basis is separable. Prove that a metric space has a countable basis if and only if it is separable.

Exercise. There are non-metrizable separable compact Hausdorff spaces! Prove that X is such a space, where

$$X = \{f : [0,1] \to [0,1] \mid x \leq y \Rightarrow f(x) \leq f(y)\}$$

is endowed with a relative topology. Hint: Tihonov's Theorem.

12 Algebras of Lipschitz functions

This section is devoted to metric properties, not merely metrizability. We shall study how to recover the metric space structure from a normed algebra of Lipschitz functions in the spirit of the Gelfand theory of commutative Banach algebras. In the sequel, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Lipschitz mappings. Let $(X, d_X), (Y, d_Y)$ be metric spaces; often we drop the subscripts from metrics, i.e. write d for both d_X and d_Y without confusion. A mapping $f : X \to Y$ is called *Lipschitz* if

$$\exists C < \infty \ \forall x, y \in X : \ d_Y(f(x), f(y)) \leq C \ d_X(x, y);$$

then the *Lipschitz constant of* f is

$$\begin{aligned} L(f) &:= \inf\{C \in \mathbb{R} \mid \forall x, y \in X : \ d(f(x), f(y)) \leq C \ d(x, y)\} \\ &= \sup_{x,y \in X: \ x \neq y} \frac{d(f(x), f(y))}{d(x, y)}. \end{aligned}$$

A mapping $f : X \to Y$ is called *bi-Lipschitz* (or a *quasi-isometry*) if it is bijective and f, f^{-1} are both Lipschitz.

Examples.

1. Lipschitz mappings are uniformly continuous, but not the vice versa: for instance, $(t \mapsto \sqrt{t}) : [0, 1] \to \mathbb{R}$ is uniformly continuous, but not Lipschitz.

2. The distance from $x \in X$ to a non-empty set $A \subset X$ is defined by

   $$d(x, A) = d(A, x) := \inf_{a \in A} d(x, a),$$

   Then $d_A = (x \mapsto d(x, A)) : X \to \mathbb{R}$ is a Lipschitz mapping, $L(d_A) \leq 1$; notice that $d_A(x) = 0$ if and only if $x \in \overline{A}$. Thus there are plenty of Lipschitz functions on a metric space.

Exercise*. Let $A, B \subset X$ be non-empty sets. Assume that the distance between A, B is positive, i.e. $d(A, B) > 0$, where

$$d(A, B) := \inf_{a \in A, \ b \in B} d(a, b).$$

Show that there exists a Lipschitz function $f : X \to \mathbb{R}$ such that

$$0 \leq f \leq 1, \quad f(A) = \{0\}, \quad f(B) = \{1\}.$$

This is the Lipschitz analogy of Urysohn's Lemma.

Tietze's Extension Theorem (Lipschitz analogy). *Let X be a metric space, $A \subset X$ non-empty, and $f : A \to \mathbb{K}$ bounded. Then there exists $F : X \to \mathbb{K}$ such that*

$$F|_A = f, \quad \|F\|_{C(X)} = \|f\|_{C(A)}, \quad \begin{cases} L(F) = L(f), & \text{when } \mathbb{K} = \mathbb{R}, \\ L(F) \leq \sqrt{2}\ L(f), & \text{when } \mathbb{K} = \mathbb{C}. \end{cases}$$

Proof. Here $\|f\|_{C(A)} := \sup_{x \in A} |f(x)|$. When $L(f) = \infty$, define $F : X \to \mathbb{K}$ by $F|_A = f$, $F(X \setminus A) = \{0\}$. For the rest of the proof, suppose $L(f) < \infty$.

Let us start with the case $\mathbb{K} = \mathbb{R}$. Define $G : X \to \mathbb{R}$ by

$$G(x) = \inf_{a \in A} (f(a) + L(f)\ d(x, a)),$$

so that $G|_A = f$, as the reader may verify. Define $F : X \to \mathbb{R}$ by

$$F(x) = \begin{cases} G(x), & \text{when } |G(x)| \leq \|f\|_{C(A)}, \\ \|f\|_{C(A)}\ \frac{G(x)}{|G(x)|}, & \text{when } |G(x)| > \|f\|_{C(A)}. \end{cases}$$

Clearly $F|_A = f$, $\|F\|_{C(X)} = \|f\|_{C(A)}$, and $L(f) \leq L(F) \leq L(G)$; let us then show that $L(G) = L(f)$. Suppose $x, y \in X$. Take $\varepsilon > 0$. Choose $a_\varepsilon \in A$ such that $G(y) \geq f(a_\varepsilon) + L(f)\ d(y, a_\varepsilon) - \varepsilon$. Then

$$\begin{aligned} G(x) - G(y) &= \inf_{a \in A} (f(a) + L(f)\ d(x, a)) - G(y) \\ &\leq (f(a_\varepsilon) + L(f)\ d(x, a_\varepsilon)) - (f(a_\varepsilon) + L(f)\ d(y, a_\varepsilon) - \varepsilon) \\ &= L(f)\ (d(x, a_\varepsilon) - d(y, a_\varepsilon)) + \varepsilon \\ &\leq L(f)\ d(x, y) + \varepsilon, \end{aligned}$$

which yields $G(x) - G(y) \leq L(f)\ d(x, y)$. Symmetrically, $G(y) - G(x) \leq L(f)\ d(x, y)$, so that $|G(x) - G(y)| \leq L(f)\ d(x, y)$. Hence we have proven that $L(G) \leq L(f)$, which completes the proof of the case $\mathbb{K} = \mathbb{R}$.

Let us consider the case $\mathbb{K} = \mathbb{C}$. Let $f_1 = \Re(f)$, $f_2 = \Im(f)$. Then using the $\mathbb{R}$-result we can extend $f_1, f_2 : A \to \mathbb{R}$ to functions $F_1, F_2 : X \to \mathbb{R}$ satisfying

$$F_j|_A = f_j, \quad L(F_j) = L(f_j) \leq L(f), \quad \|F_j\|_{C(X)} = \|f_j\|_{C(A)}.$$

Let us define $G : X \to \mathbb{C}$ by $G = F_1 + iF_2$, and define $F : X \to \mathbb{C}$ by

$$F(x) = \begin{cases} G(x), & \text{when } |G(x)| \leq \|f\|_{C(A)}, \\ \|f\|_{C(A)}\ \frac{G(x)}{|G(x)|}, & \text{when } |G(x)| > \|f\|_{C(A)}. \end{cases}$$

Then $\|F\|_{C(X)} = \|f\|_{C(A)}$. Moreover, we obtain $L(G) \leq \sqrt{2}\ L(f)$, because $|z| \leq \sqrt{2}\ \max\{|\Re(z)|, |\Im(z)|\}$ for every $z \in \mathbb{C}$; hence

$$L(f) \leq L(F) \leq L(G) \leq \sqrt{2}\ L(f),$$

completing the proof □

Lipschitz spaces. Let X be a metric space. Let

$$\mathrm{Lip}(X) = \mathrm{Lip}(X, \mathbb{K}) := \{f : X \to \mathbb{K} :\ \|f\|_{\mathrm{Lip}} = \max(\|f\|_{C(X)}, L(f)) < \infty\}.$$

A *pointed metric space* is a metric space X with a distinguished element, the *base point* $e_X = e \in X$; let

$$\mathrm{Lip}_0(X) = \mathrm{Lip}_0(X, \mathbb{K}) := \{f : X \to \mathbb{K} \mid f(e) = 0,\ L(f) < \infty\}.$$

Notice that if the diameter $\mathrm{diam}(X) = \sup_{x,y \in X} d(x,y)$ of the space is finite then $\mathrm{Lip}_0(X)$ is contained in $\mathrm{Lip}(X)$,

Exercise*. Show that $\mathrm{Lip}(X)$ is a Banach space with the norm $f \mapsto \|f\|_{\mathrm{Lip}}$. Show that $\mathrm{Lip}_0(X)$ is a Banach space with the norm $f \mapsto L(f)$. Show that these spaces are topological algebras if $\mathrm{diam}(X) < \infty$.

Arens–Eells space. Let X be a metric space, $x, y \in X$. The *xy-atom* is the function $m_{xy} : X \to \mathbb{K}$ defined by

$$m_{xy}(x) = 1, \quad m_{xy}(y) = -1, \quad m_{xy}(z) = 0 \text{ otherwise.}$$

A *molecule on* X is a linear combination $m = \sum_{j=1}^{n} a_j\ m_{x_j y_j}$ of such atoms; then $\{x \in X \mid m(x) \neq 0\}$ is a finite set and $\sum_{x \in X} m(x) = 0$. Let M denote the $\mathbb{K}$-vector space of the molecules on X. Notice that a molecule may have several representations as a linear combination of atoms. Let us define a mapping $m \mapsto \|m\|_{AE} : M \to \mathbb{R}$ by

$$\|m\|_{AE} := \inf\left\{\sum_{j=1}^{n} |a_j|\ d(x_j, y_j) :\ n \in \mathbb{Z}^+,\ m = \sum_{j=1}^{n} a_j\ m_{x_j y_j}\right\};$$

obviously this is a seminorm on the space of the molecules, but we shall prove that it is actually a norm; for the time being, we have to define the *Arens–Eells space* $AE(X)$ for X by completing the vector space M with respect to the Arens–Eells-**seminorm** $m \mapsto \|m\|_{AE}$ modulo the subspace $\{v :\ \|v\|_{AE} = 0\}$.

Theorem. *The Banach space dual of $AE(X)$ is isometrically isomorphic to* $\mathrm{Lip}_0(X)$.

Proof. Let us define two linear mappings $T_1 : AE(X)' \to \mathrm{Lip}_0(X)$ and $T_2 : \mathrm{Lip}_0(X) \to AE(X)'$ by

$$(T_1\phi)(x) := \phi(m_{xe}), \quad (T_2 f)(m) := \sum_{y \in X} f(y)\ m(y),$$

where $e \in X$ is the base point, and $m \in M$ is a molecule (so that $T_2 f$ is uniquely extended to a linear functional on $AE(X)$). These definitions are sound indeed: Firstly,

$$(T_1\phi)(e) = \phi(m_{ee}) = \phi(0) = 0,$$

$$\begin{aligned} |(T_1\phi)(x) - (T_1\phi)(y)| &= |\phi(m_{xe} - m_{ye})| = |\phi(m_{xy})| \leq \|\phi\|\ \|m_{xy}\|_{AE} \\ &\leq \|\phi\|\ d(x,y), \end{aligned}$$

so that $T_1\phi \in \mathrm{Lip}_0(X)$ and $L(T_1\phi) \leq \|\phi\|$; we have even proven that $T_1 \in \mathcal{L}(AE(X)', \mathrm{Lip}_0(X))$ with norm $\|T_1\| \leq 1$. Secondly, let $\varepsilon > 0$ and $m \in M$. We may choose $(a_j)_{j=1}^n \subset \mathbb{K}$ and $((x_j,y_j))_{j=1}^n \subset X \times X$ such that

$$m = \sum_{j=1}^n a_j\ m_{x_j y_j}, \quad \sum_{j=1}^n |a_j|\ d(x_j,y_j) \leq \|m\|_{AE} + \varepsilon.$$

Then

$$\begin{aligned} |(T_2 f)(m)| &= \left|(T_2 f)\ \sum_{j=1}^n a_j\ m_{x_j y_j}\right| = \left|\sum_{j=1}^n a_j\ (f(x_j) - f(y_j))\right| \\ &\leq \sum_{j=1}^n |a_j|\ |f(x_j) - f(y_j)| \\ &\leq L(f)\ \sum_{j=1}^n |a_j|\ d(x_j,y_j) \\ &\leq L(f)\ (\|m\|_{AE} + \varepsilon), \end{aligned}$$

meaning that $T_2 \in \mathcal{L}(\mathrm{Lip}_0(X), AE(X)')$ with norm $\|T_2\| \leq 1$. Next we notice that $T_2 = T_1^{-1}$:

$$(T_1(T_2 f))(x) = (T_2 f)(m_{xe}) = \sum_{y \in X} f(y)\ m_{xe}(y) = f(x) - f(e) = f(x),$$

$$\begin{aligned}(T_2(T_1\phi))(m) &= \sum_{y\in X}(T_1\phi)(y)\ m(y) \ = \ \sum_{y\in X}\phi(m_{ye})\ m(y)\\ &= \phi\left(\sum_{y\in X} m(y)\ m_{ye}\right) = \phi(m).\end{aligned}$$

Finally, for $f \in \mathrm{Lip}_0(X)$ we have

$$L(f) = L(T_1T_2f) \le \|T_1\|\ \|T_2f\| \le \|T_2f\| \le \|T_2\|L(f) \le L(f),$$

so that $T_2, T_1 = T_2^{-1}$ are isometries □

Remark. Let us denote

$$((f,m)\mapsto\langle f,m\rangle) : \mathrm{Lip}_0(X)\times AE(X)\to\mathbb{K}$$

where

$$\langle f,m\rangle = \sum_{x\in X} f(x)\ m(x)$$

if $m \in M$. From now on, the weak*-topology of $\mathrm{Lip}_0(X)$ refers to the $AE(X)$-induced topology, with the interpretation

$$AE(X) \subset AE(X)'' \cong \mathrm{Lip}_0(X)'.$$

Next we show how X is canonically embedded in the Arens–Eells space:

Corollary. *The Arens–Eells seminorm $m \mapsto \|m\|_{AE}$ is a norm, and the mapping $(x \mapsto m_{xe}) : X \to AE(X)$ is an isometry.*

Proof. Take $m \in M$, $m \neq 0$. Choose $x_0 \in X$ such that $m(x_0) \neq 0$. Due to the theorem above,

$$\|m\|_{AE} \overset{\text{Hahn}-\text{Banach}}{=} \sup_{f\in AE(X)':\|f\|\le 1}|\langle f,m\rangle| = \sup_{f\in\mathrm{Lip}_0(X):L(f)\le 1}\left|\sum_{x\in X} f(x)\ m(x)\right|.$$

Let $A := \{e\}\cup\{x\in X \mid m(x)\neq 0\}$. Let $r := d(x_0, A\backslash\{x_0\})$. By the Lipschitz analogy of Tietze's Extension Theorem, there exists $f_0 \in \mathrm{Lip}_0(X,\mathbb{R})$ such that $f_0(x_0) = r > 0$, $f_0(A\setminus\{x_0\}) = \{0\}$, and $L(f_0) = 1$. Thereby

$$\|m\|_{AE} \ge |\langle f_0,m\rangle| = \left|\sum_{x\in X} f_0(x)\ m(x)\right| = |f_0(x_0)\ m(x_0)| > 0,$$

i.e. $m \mapsto \|m\|_{AE}$ is actually a norm.

Let $x, y \in X$. Clearly $\|m_{xy}\|_{AE} \le d(x,y)$. Define $\widetilde{d_y}(z) := d(z,y) - d(e,y)$, where $e \in X$ is the base point. Now $\widetilde{d_y} \in \mathrm{Lip}_0(X)$ and $L(\widetilde{d_y}) = 1$, so that

$$\begin{aligned} \|m_{xy}\|_{AE} &\ge |\langle \widetilde{d_y}, m_{xy}\rangle| = \left|\sum_{z\in X} \widetilde{d_y}(z)\ m_{xy}(z)\right| = |\widetilde{d_y}(x) - \widetilde{d_y}(y)| \\ &= d(x,y). \end{aligned}$$

Hence $\|m_{xe} - m_{ye}\|_{AE} = \|m_{xy}\|_{AE} = d(x,y)$ □

Nets and convergence. A partial order $(J, \le)$ is called a *directed* if

$$\forall i, j \in J\ \exists k \in J: \ i \le k, \ j \le k.$$

A *net* in a topological space (X, τ) is a family $(x_j)_{j\in J} \subset X$, where $J = (J, \le)$ is directed. A net $(x_j)_{j\in J} \subset X$ *converges to a point* $x \in X$, denoted by

$$x_j \to x \quad \text{or} \quad x_j \to_{j\in J} x \quad \text{or} \quad x = \lim x_j = \lim_{j\in J} x_j,$$

if for every $U \in \mathcal{V}_\tau(x)$ there exists $j_U \in J$ such that $x_j \in U$ whenever $j_U \le j$.

An example of a net is a *sequence* $(x_n)_{n\in\mathbb{N}} \subset X$, where $\mathbb{N}$ has the usual partial order; sequences characterize topology in spaces of countable local bases, for instance metric spaces. But there are more complicated topologies, where sequences are not enough; for example, weak*-topology for infinite-dimensional spaces.

Exercise*. Nets can be used to characterize the topology: Let (X, τ) be a topological space and $A \subset X$. Show that $x \in \overline{A} \subset X$ if and only if there exists a net $(x_j)_{j\in J} \subset A$ such that $x_j \to x$. Let $f : X \to Y$; show that $f \in C(X,Y)$ if and only if $x_j \to x \in X \ \Rightarrow f(x_j) \to f(x) \in Y$.
(Hint: define a partial order relation on $\mathcal{V}_\tau(x)$ by $U \le V \ \Leftrightarrow \ V \subset U$.)

Lemma. *Let E be a Banach space. The weak*-converging nets in E' are bounded.*

Proof. Let $f_j \to f$ in the weak*-topology of E', i.e. $\langle f_j, \phi\rangle \to \langle f, \phi\rangle \in \mathbb{K}$ for every $\phi \in E$. Define $T_j : E \to \mathbb{K}$ by $\phi \mapsto \langle f_j, \phi\rangle$. Since $T_j\phi \to \langle f, \phi\rangle \in \mathbb{K}$, we have $\sup_{j\in J} |T_j\phi| < \infty$ for every $\phi \in E$, so that $C := \sup_{j\in J} \|T_j\| < \infty$ according to the Banach–Steinhaus Theorem. Thereby

$$\|f_j\| \overset{\text{Hahn−Banach}}{=} \sup_{\phi\in E:\|\phi\|\le 1} |\langle f_j, \phi\rangle| = \sup_{\phi\in E:\|\phi\|\le 1} |T_j\phi| \overset{\text{Hahn−Banach}}{=} \|T_j\| \le C,$$

so that the net $(f_j)_{j\in J} \subset E'$ is bounded □

Proposition. *On bounded subsets of* $\mathrm{Lip}_0(X)$ *the weak*-topology is the topology of pointwise convergence. Moreover, if* X *is compact, on bounded sets these topologies coincide with the topology of uniform convergence.*

Proof. Let $\mathcal{E} \subset \mathrm{Lip}_0(X)$ be a bounded set containing a net $(f_j)_{j\in J}$ such that $f_j \to f$ in the weak*-topology. Endow the norm-closure $\overline{\mathcal{E}}$ with the relative weak*-topology τ_1, and also with the topology τ_2 of pointwise convergence. If $x \in X$ then

$$f_j(x) = f_j(x) - f_j(e) = \langle f_j, m_{xe}\rangle \to \langle f, m_{xe}\rangle = f(x) - f(e) = f(x),$$

i.e. $f_j \to f$ pointwise. This means that the topology of pointwise convergence is weaker than the weak*-topology, $\tau_2 \subset \tau_1$. Now τ_1 is compact due to the Banach–Alaoglu Theorem, and of course τ_2 is Hausdorff; hence $\tau_1 = \tau_2$, the weak*-topology and the topology of the pointwise convergence coincide on bounded subsets.

Now suppose X is a compact metric space. Uniform convergence trivially implies pointwise convergence. Let $(f_j)_{j\in J} \subset \mathcal{E}$ be as above, $f_j \to f$ pointwise. Since $\mathcal{E}$ is bounded, there exists $C < \infty$ such that $L(g) \le C$ for every $g \in \mathcal{E}$. It is easy to check that $L(f) \le C$. Take $\varepsilon > 0$. Since X is compact, there exists $\{x_k\}_{k=1}^{n_\varepsilon} \subset X$ such that

$$\forall x \in X \; \exists k \in \{1, \cdots, n_\varepsilon\} : \; d(x, x_k) < \varepsilon.$$

Due to the pointwise convergence $f_j \to f$, there exists $j_\varepsilon \in J$ such that

$$|f_j(x_k) - f(x_k)| < \varepsilon$$

for every $k \in \{1, \cdots, n_\varepsilon\}$ whenever $j_\varepsilon \le j$. Take $x \in X$. Take $k \in \{1, \cdots, n_\varepsilon\}$ such that $d(x, x_k) < \varepsilon$. Then

$$\begin{aligned} |f_j(x) - f(x)| &\le |f_j(x) - f_j(x_k)| + |f_j(x_k) - f(x_k)| + |f(x_k) - f(x)| \\ &\le L(f_j)\; d(x, x_k) + \varepsilon + L(f)\; d(x_k, x) \\ &\le C\,\varepsilon + \varepsilon + C\,\varepsilon \;=\; (2C+1)\,\varepsilon. \end{aligned}$$

Thereby $\|f_j - f\|_{C(X)} \to 0$; pointwise convergence on bounded subsets implies uniform convergence, when X is compact □

Algebra $\mathrm{Lip}_0(X)$**.** Let X be a metric space such that $\mathrm{diam}(X) < \infty$. In the sequel, we shall call $\mathrm{Lip}_0(X)$ an *algebra*, even though $\mathbb{I} \notin \mathrm{Lip}_0(X)$. An *algebra homomorphism* between such non-unital algebras is a linear and multiplicative mapping; then even the 0-mapping is a homomorphism!

Proposition. *Let X, Y be metric spaces with finite diameters, with the respective base points e_X, e_Y. Let $g : Y \to X$ be a Lipschitz mapping such that $g(e_Y) = e_X$. Then the mapping*

$$L_g : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y), \quad f \mapsto f \circ g,$$

is an algebra homomorphism, and $\|L_g\| = L(g)$.

Proof. If $f \in \mathrm{Lip}_0(X)$ and $x, y \in Y$ then

$$|f(g(x)) - f(g(y))| \leq L(f)\ d(g(x), g(y)) \leq L(f)\ L(g)\ d(x, y).$$

Hence $L(L_g f) = L(f \circ g) \leq L(f)\ L(g)$, implying $\|L_g\| \leq L(g)$. Take $y_0 \in Y$. Define $f_0 \in \mathrm{Lip}_0(X)$ by $f_0(x) := d(x, g(y_0)) - d(e_X, g(y_0))$, so that $L(f_0) = 1$. Take $y \in Y$, $y \neq y_0$. Then

$$\begin{aligned} \|L_g\| &\geq L(L_g(f_0)) \\ &\geq \frac{|(L_g f_0)(y) - (L_g f_0)(y_0)|}{d(y, y_0)} = \frac{d(g(y), g(y_0))}{d(y, y_0)}, \end{aligned}$$

so that $\|L_g\| \geq L(g)$; hence $\|L_g\| = L(g)$.

If $\lambda \in \mathbb{K}$ and $f, h \in \mathrm{Lip}_0(X)$ then

$$\begin{aligned} L_g(\lambda f) &= (\lambda f) \circ g = \lambda\ (f \circ g) = \lambda\ L_g f, \\ L_g(f + h) &= (f + h) \circ g = f \circ g + h \circ g = L_g f + L_g h, \\ L_g(fh) &= (fh) \circ g = (f \circ g)(h \circ g) = (L_g f)(L_g h), \end{aligned}$$

so that L_g is a homomorphism □

Order-completeness. Non-empty $\mathcal{B} \subset \mathrm{Lip}(X, \mathbb{R})$ is called *order-complete* if

$$\sup \mathcal{G}, \inf \mathcal{G} \in \mathcal{B}$$

for every bounded family $\mathcal{G} \subset \mathcal{B}$. Here supremums and infimums are pointwise, naturally.

Uniform separation. A family $\mathcal{F} \subset \mathrm{Lip}_0(X)$ *separates uniformly the points of* X if

$$\exists C < \infty\ \forall x, y \in X\ \exists g \in \mathcal{F} :\ L(g) \leq C,\ |g(x) - g(y)| = d(x, y).$$

In a striking resemblance with the "classical" Stone–Weierstrass Theorem, we have the following:

Theorem (Lipschitz Stone–Weierstrass). *Let X be a compact metric space. Let $\mathcal{A}$ be an involutive, weak*-closed subalgebra of* $\mathrm{Lip}_0(X)$ *separating the points of X uniformly. Then* $\mathcal{A} = \mathrm{Lip}_0(X)$.

Proof. As in the proof of the "classical" Stone–Weierstrass Theorem, involutivity justifies our concentration on the $\mathbb{R}$-scalar case, where the involution is trivial, $f^* = f$. Hence we assume that $\mathcal{A}$ is a weak*-closed $\mathbb{R}$-subalgebra of $\mathrm{Lip}_0(X, \mathbb{R})$ separating the points of X uniformly.

Let us show that $\mathcal{B} = \mathcal{A} + \mathbb{R}\mathbb{I}$ is closed under the pointwise convergence of bounded nets. Let $(g_j)_{j\in J} \subset \mathcal{B}$ be a bounded net converging to pointwise to $g \in \mathrm{Lip}(X)$; here $g_j = f_j + \lambda_j \mathbb{I}$ with $f_j \in \mathcal{A}$ and $\lambda_j \in \mathbb{R}$. Especially

$$\lambda_j = f_j(e) + \lambda_j = g_j(e) \to g(e) \in \mathbb{R}.$$

Thus

$$f_j(x) = g_j(x) - \lambda_j \to g(x) - g(e) \in \mathbb{R},$$

i.e. $f_j \to g - g(e)\mathbb{I}$ pointwise. But $(f_j)_{j\in J} \subset \mathcal{A}$ is a bounded net, so that $f_j \to g - g(e)\mathbb{I}$ in the weak*-topology; since $\mathcal{A}$ is weak*-closed, $g - g(e)\mathbb{I} \in \mathcal{A}$. Thereby

$$g = (g - g(e)\mathbb{I}) + g(e)\mathbb{I} \in \mathcal{A} + \mathbb{R}\mathbb{I} = \mathcal{B};$$

$\mathcal{B}$ is closed under the pointwise convergence of bounded nets.

Let us show that $\mathcal{B}$ is order-complete. First, let $g \in \mathcal{B}$. Take $\varepsilon > 0$. Let

$$g_\varepsilon(x) := \sqrt{g(x)^2 + \varepsilon^2}.$$

By the Weierstrass Approximation Theorem, there exists a sequence $(P_{\varepsilon n})_{n=1}^\infty$ of real-valued polynomials such that $P_{\varepsilon n}(0) = \varepsilon$ and

$$P'_{\varepsilon n}(t) \to_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{t^2 + \varepsilon^2} = \frac{t}{\sqrt{t^2 + \varepsilon^2}}$$

uniformly on $[-\|g\|_{C(X)}, \|g\|_{C(X)}]$; consequently, $(P_{\varepsilon n}(g))_{n=1}^\infty \subset \mathcal{B}$ is a bounded sequence, converging uniformly to g_ε; hence $P_{\varepsilon n}(g) \to g_\varepsilon$ also pointwise. Since $\mathcal{B}$ is closed under the pointwise convergence of bounded nets, we deduce $g_\varepsilon \in \mathcal{B}$; consequently, $(g_\varepsilon)_{0<\varepsilon\leq 1}$ is a bounded net in $\mathcal{B}$, so that $g_0 := \lim_{\varepsilon\to 0^+} g_\varepsilon$ belongs to $\mathcal{B}$. But $g_0(x) = |g(x)|$, so that $g \in \mathcal{B}$ implies $|g| \in \mathcal{B}$. Therefore if $f, g \in \mathcal{B}$ then

$$\max(f, g) = \frac{f+g}{2} + \frac{|f+g|}{2}, \quad \min(f, g) = \frac{f+g}{2} - \frac{|f+g|}{2}$$

belong to $\mathcal{B}$. Let $\mathcal{G} \subset \mathcal{B}$ be a bounded non-empty family. Let $\mathcal{H} \subset \mathrm{Lip}(X, \mathbb{R})$ be the smallest family closed under taking maximums and minimums and containing $\mathcal{G}$. Now $\mathcal{H} \subset \mathcal{B}$, since $\mathcal{B}$ is closed under taking maximums and minimums. Moreover, $\mathcal{H}$ is bounded. Clearly

$$\sup \mathcal{G} = \sup \mathcal{H} \in \mathrm{Lip}(X, \mathbb{R}) \quad \text{and} \quad \inf \mathcal{G} = \inf \mathcal{H} \in \mathrm{Lip}(X, \mathbb{R}).$$

Let $g := \sup \mathcal{G} \in \mathrm{Lip}(X, \mathbb{R})$. Take $\varepsilon > 0$. For each $x \in X$ there exists $g_x \in \mathcal{G}$ such that $g(x) - \varepsilon < g_x(x)$. Due to the continuity of g_x, there exists $U_x \in \mathcal{V}(x)$ such that $g(y) - \varepsilon < g_x(y)$ for every $y \in U_x$. Then $\{U_x \mid x \in X\}$ is an open cover of the compact space X, so that there is a finite subcover $\{U_{x_j} \mid 1 \le j \le n\}$. Let $h_\varepsilon := \max(g_{x_1}, \cdots, g_{x_n}) \in \mathcal{H}$. Then

$$g(x) - \varepsilon < h_\varepsilon(x) < g(x)$$

for every $x \in X$, so that $(h_\varepsilon)_{0<\varepsilon\le 1} \subset \mathcal{H} \subset \mathcal{B}$ is a bounded net, $h_\varepsilon \to_{\varepsilon \to 0^+} g$. Hence $\sup \mathcal{G} = g \in \mathcal{B}$, because $\mathcal{B}$ is closed under the pointwise convergence of bounded nets. Similarly one proves that $\inf \mathcal{G} \in \mathcal{B}$. Thus $\mathcal{B}$ is order-complete.

Take $f \in \mathrm{Lip}_0(X, \mathbb{R})$. We have to show that $f \in \mathcal{A}$. We may assume that $L(f) \le 1$. Due to the uniform separation, for every $x, y \in X$ there exists $g_{xy} \in \mathcal{A}$ such that $L(g_{xy}) \le C$ (C does not depend on $x, y \in X$) and $|g_{xy}(x) - g_{xy}(y)| = d(x, y)$. Since $|f(x) - f(y)| \le L(f)\ d(x, y) \le d(x, y)$ and since $\mathcal{A}$ is an algebra, there exists $h_{xy} \in \mathcal{A}$ satisfying $h_{xy}(x) - h_{xy}(y) = f(x) - f(y)$ and $L(h_{xy}) \le C$. Define $f_{xy} \in \mathcal{B}$ by

$$f_{xy} := h_{xy} - (h_{xy}(y) - f(y))\mathbb{I}.$$

Then $f_{xy}(x) = f(x)$ and $f_{xy}(y) = f(y)$, $L(f_{xy}) = L(h_{xy}) \le C$, and

$$\|f_{xy}\|_{C(X)} \le \|h_{xy}\|_{C(X)} + |h_{xy}(y)| + |f(y)| \le (2C + 1)\ r(X),$$

where $r(X) := \sup_{z \in X} d(z, e) < \infty$ is the "radius" of the space. The family $(f_{xy})_{x,y \in X} \subset \mathcal{B}$ is hence bounded; due to the order-completeness of $\mathcal{B}$,

$$f = \inf_{x \in X} \sup_{y \in X} f_{xy}$$

belongs to $\mathcal{B}$; but $f(e) = 0$, so that $f \in \mathcal{A}$ □

Quotient metrics. Let X be a compact metric space and $\mathcal{A} \subset \mathrm{Lip}_0(X)$ be an involutive, weak*-closed subalgebra. Let $R_\mathcal{A}$ be the equivalence relation

$$(x, y) \in R_\mathcal{A} \overset{\text{definition}}{\iff} \forall f \in \mathcal{A} : \ f(x) = f(y).$$

Let $[x] := \{y \in X \mid (x, y) \in R_\mathcal{A}\}$. Let us endow $X_\mathcal{A} := X/R_\mathcal{A} = \{[x] \mid x \in X\}$ with the metric

$$d_{X_\mathcal{A}}([x], [y]) := \sup_{f \in \mathcal{A}: L(f) \le 1} |f(x) - f(y)|.$$

Let $\pi = (x \mapsto [x]) : X \to X_\mathcal{A}$. Recall that this induces a homomorphism $L_\pi = (\tilde{f} \mapsto \tilde{f} \circ \pi) : \mathrm{Lip}_0(X_\mathcal{A}) \to \mathrm{Lip}_0(X)$.

Corollary. *Let X be a compact metric space, and let $\mathcal{A}$ be an involutive, weak*-closed subalgebra of* $\mathrm{Lip}_0(X)$*. Then* $L_\pi : \mathrm{Lip}_0(X_\mathcal{A}) \to \mathcal{A} \subset \mathrm{Lip}_0(X)$ *is a bounded algebra isomorphism* $\mathrm{Lip}_0(X_\mathcal{A}) \cong \mathcal{A}$ *with a bounded inverse.*

Exercise*. Prove the previous Corollary.

Exercise*. Show that weak*-closed ideals of $\mathrm{Lip}_0(X)$ are involutive, when X is compact. (Hint: Lipschitz–Stone–Weierstrass.)

Varieties and ideals. Let X be a metric space, $S \subset X$, and $\mathcal{J} \subset \mathrm{Lip}_0(X)$. Then

$$\mathcal{I}(S) := \{f \in \mathrm{Lip}_0(X) \mid \forall x \in S : \ f(x) = 0\}$$

is a weak*-closed ideal of $\mathrm{Lip}_0(X)$ (the *ideal of* S), and

$$V(\mathcal{J}) := \{x \in X \mid \forall f \in \mathcal{J} : \ f(x) = 0\}$$

is a closed subset of X (the *variety of* $\mathcal{J}$).

Theorem. *Let X be a compact metric space, $\mathcal{J}$ be a weak*-closed ideal of* $\mathrm{Lip}_0(X)$*. Then* $\mathcal{J} = \mathcal{I}(V(\mathcal{J}))$.

Exercise*. Prove the previous theorem. (Hint: show that $d(x, V(\mathcal{J})) = d_{X_\mathcal{J}}([x], V(\mathcal{J}))$ for every $x \in X$, use Lipschitz–Stone–Weierstrass.)

Corollary. *Let X be a compact metric space, and let* $\omega : \mathrm{Lip}_0(X) \to \mathbb{K}$ *be an algebra homomorphism. Then ω is weak*-continuous if and only if* $\omega = \omega_x := (x \mapsto f(x))$ *for some $x \in X$.*

Proof. If $\omega_x := (x \mapsto f(x)) : \mathrm{Lip}_0(X) \to \mathbb{K}$ then $\omega_x = m_{xe} \in AE(X)$ in the sense that $\langle f, \omega_x \rangle = f(x) = \langle f, m_{xe} \rangle$; hence evaluation homomorphisms are weak*-continuous.

Conversely, let $\omega : \mathrm{Lip}_0(X) \to \mathbb{K}$ be a weak*-continuous homomorphism. Then $\mathrm{Ker}(\omega)$ is an weak*-closed ideal of $\mathrm{Lip}_0(X)$, hence involutive. Thus by the previous Theorem $\mathrm{Ker}(\omega) = \mathcal{I}(V)$ for some $V \subset X$. Notice that $0 = \omega_e$; assume that $\omega \neq 0$. Since ω is a surjective linear mapping onto $\mathbb{K}$, $\mathrm{Ker}(\omega)$ must be of co-dimension 1 in $\mathrm{Lip}_0(X)$, and thereby $V = \{e, x\}$ for some $x \in X$. Hence $\omega = (f \mapsto \lambda f(x))$ for some $\lambda \in \mathbb{K}$, $\lambda \neq 0$. Choose $f \in \mathrm{Lip}_0(X)$ such that $f(x) = 1$, so that

$$\lambda = \omega(f) = \omega(f^2) = \omega(f)^2 = \lambda^2.$$

This yields $\lambda = 1$, i.e. $\omega = \omega_x := (f \mapsto f(x))$ □

Spectra. In these lecture notes we started with **unital** algebras (which we simply called "algebras"). At the present, we have encountered **non-unital** algebras, e.g. $\mathrm{Lip}_0(X)$ and its ideals on a compact metric space X. In the sequel, let the word "algebra" stand for both unital and non-unital algebras. We say that a *homomorphism* is a linear multiplicative mapping between algebras such that if both algebras are unital then one unit element is mapped to another; the set of homomorphisms $\mathcal{A} \to \mathcal{B}$ is denoted by $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$. Notice that $0 \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ if and only if $\mathcal{A}$ or $\mathcal{B}$ is non-unital. With these nominations, let the *spectrum* of a Banach space and a commutative topological algebra $\mathcal{A}$ be

$$\mathrm{Spec}(\mathcal{A}) := \mathrm{Hom}(\mathcal{A}, \mathbb{K}).$$

If furthermore $\mathcal{A} \cong E'$ for a Banach space E, let

$$\mathrm{Spec}^{w^*}(\mathcal{A}) := \{\omega \in \mathrm{Spec}(\mathcal{A}) \mid \omega \text{ is weak}^*\text{−continuous}\}.$$

Endow all these spectra with the metric given by the norm of the Banach space $\mathcal{A}'$; there are also the relative weak*-topologies of $\mathcal{A}'$ on the spectra.

Theorem. *Let X be a compact metric space. Then the metric topology and the relative weak*-topology of* $\mathrm{Spec}^{w^*}(\mathrm{Lip}_0(X))$ *are the same, and X is isometric to* $\mathrm{Spec}^{w^*}(\mathrm{Lip}_0(X))$. *Moreover,* $\mathrm{Spec}^{w^*}(\mathrm{Lip}_0(X)) = \mathrm{Spec}(\mathrm{Lip}_0(X))$.

Proof. Let us denote $\mathcal{A} := \mathrm{Lip}_0(X)$. The weak*-topology on $K := \mathrm{Spec}^{w^*}(\mathcal{A})$ is the topology induced by the family $\{\widehat{f} \mid f \in \mathcal{A}\}$, where $\widehat{f} : K \to \mathbb{K}$ is defined by $\widehat{f}(\omega) := \omega(f)$ (sort of Gelfand transform).

The previous Corollary indicates that K is the set of evaluation homomorphisms $\omega_x = (f \mapsto f(x))$, and we know that

$$\iota = (x \mapsto m_{xe} = \omega_x) : X \mapsto AE(X) \subset \mathcal{A}'$$

is an isometry. Hence X is isometric to K.

The norm topology of $\mathcal{A}'$ is stronger than the weak*-topology, so that the metric topology on K is stronger than the relative weak*-topology. Notice that $f_y \in \mathcal{A}$, where $\widehat{f_y}(\omega_x) = f_y(x) = d(x,y) - d(e,y)$; hence $\widehat{f_y} : K \to \mathbb{R}$ is weak*-continuous on K, so that $\widehat{f_y}^{-1}(U) \subset K$ is weak*-open for every open set $U \subset \mathbb{R}$. Thus the metric ball

$$\begin{aligned} B(\omega_y, \varepsilon) &= \{\omega_x : \|\omega_x - \omega_y\| < \varepsilon\} = \{\omega_x : d(x,y) < \varepsilon\} \\ &= \{\omega_x : \widehat{f_y}(\omega_x) < \varepsilon - d(e,y)\} \end{aligned}$$

is a weak*-open set. Clearly $\{B(\omega_y, \varepsilon) \mid y \in X,\ \varepsilon > 0\}$ is a basis for the metric topology of the spectrum; thereby the metric topology is weaker than the weak*-topology. Consequently, the topologies must be the same.

Let us extend $\omega \in \mathrm{Spec}(\mathcal{A}) \subset \mathcal{A}'$ linearly to $\widetilde{\omega} : \mathrm{Lip}(X) \to \mathbb{K}$ by setting $\widetilde{\omega}(\mathbb{I}) = 1$. Then $\widetilde{\omega} \in \mathrm{Spec}(\mathrm{Lip}(X))$. **Assume** that for every $x \in X$ there exists $f_x \in \mathrm{Ker}(\widetilde{\omega})$ such that $f_x(x) \neq 0$. Then pick a neighborhood $U_x \in \mathcal{V}(x)$ such that $0 \notin f_x(U_x)$. Due to the compactness of X we may pick a finite subcover $\{U_{x_j}\}_{j=1}^n$ out of the open cover $\{U_x \mid x \in X\}$. Then

$$f := \sum_{j=1}^n |f_{x_j}|^2 = \sum_{j=1}^n \overline{f_{x_j}}\ f_{x_j} \in \mathrm{Ker}(\widetilde{\omega});$$

so f belongs to an ideal of $\mathrm{Lip}(X)$, but on the other hand $f(x) > 0$ for every $x \in X$, so that $1/f \in \mathrm{Lip}(X)$ as the reader may verify — a **contradiction**. Hence there exists $x \in X$ such that $f(x) = 0$ for every $f \in \mathrm{Ker}(\widetilde{\omega})$. The reader may prove analogies of the Lipschitz Stone–Weierstrass Theorem and its consequences replacing (non-unital) subalgebras of $\mathrm{Lip}_0(X)$ by (unital) subalgebras of $\mathrm{Lip}(X)$; of course, the weak*-convergence has to be replaced by the pointwise convergence of bounded nets; then it follows that $\mathrm{Ker}(\widetilde{\omega}) = \{f \in \mathrm{Lip}(X) \mid f(x) = 0\}$, which would imply that $\widetilde{\omega} = (f \mapsto f(x))$.

Hence $\widetilde{\omega} = (f \mapsto f(x))$ for some $x \in X$, and consequently $\omega = \omega_x$. Evaluation homomorphisms are weak*-continuous, so that we have proven that $\mathrm{Spec}(\mathcal{A}) = K$ □

Theorem. *Let $\mathcal{A}$ be a Banach space and a non-unital commutative topological algebra, and endow* $\mathrm{Spec}(\mathcal{A})$ *with the relative metric of $\mathcal{A}'$. Then* $\mathrm{Spec}(\mathcal{A})$ *is a complete pointed metric space of finite diameter, and the extended Gelfand transform*

$$(f \mapsto \widehat{f}) : \mathcal{A} \to \mathrm{Lip}_0(\mathrm{Spec}(\mathcal{A})),$$

(where $\widehat{f}(\omega) := \omega(f)$ for $f \in \mathcal{A}$ and $\omega \in$ $\mathrm{Spec}(\mathcal{A})$*) is of norm ≤ 1.*

Proof. We may always endow $\mathcal{A}$ with an equivalent Banach algebra norm (even though the algebra is non-unital). From the Gelfand theory of commutative Banach algebras, we know that $\mathrm{Spec}(\mathcal{A})$ is a bounded weak*-closed (even weak*-compact) subset of $\mathcal{A}'$; hence the metric is complete, and the diameter is finite.

Now let $x \mapsto \|x\|$ be the original norm of $\mathcal{A}$. Let $\phi, \psi \in \mathrm{Spec}(\mathcal{A})$. Then

$$|\widehat{x}(\phi) - \widehat{x}(\psi)| = |(\phi - \psi)(x)| \leq \|\phi - \psi\|\ \|x\|,$$

so that $L(\widehat{x}) \leq \|x\|$. Notice that $\widehat{x}(0) = 0$, so that the proof is complete □

Theorem. *Let $\mathcal{A}$ be a commutative Banach algebra, and endow* $\mathrm{Spec}(\mathcal{A})$ *with the relative metric of $\mathcal{A}'$. Then* $\mathrm{Spec}(\mathcal{A})$ *is a complete metric space of diameter at most* 2*, and the Gelfand transform* $(f \mapsto \widehat{f}) : \mathcal{A} \to \mathrm{Lip}(\mathrm{Spec}(\mathcal{A}))$ *is of norm* 1.

Proof. In the Gelfand theory we have seen that $\mathrm{Spec}(\mathcal{A})$ belongs to the closed unit ball of $\mathcal{A}'$, so that the diameter of the spectrum is at most 2. If $\phi \in \mathrm{Spec}(\mathcal{A})$ and $x \in \mathcal{A}$ then $|\widehat{x}(\phi)| = |\phi(x)| \le \|x\|$, and the rest of the proof is as in the previous Theorem $\square$

Remark. Let $\mathcal{A}$ be a Banach space and a non-unital topological algebra. If $\mathrm{Spec}(\mathcal{A})$ is compact in the metric topology then the metric topology is the relative weak*-topology, and $\mathrm{Lip}_0(\mathrm{Spec}(\mathcal{A})) \subset C(\mathrm{Spec}(\mathcal{A}))$.