TKK Matematiikan laitos Arponen/Pursiainen

Mat-1.3651 Numerical Linear Algebra, spring 2008
(Numeerinen matriisilaskenta, kevät 2008)

Exercise 7 (13.3.2008)

Please hand in the exercises marked with an asterisk (*) either to the assistant's folder in front of U313 or latest at the beginning of the exercise.

* 1. Given $A \in \mathbb{C}^{m\times n}$ of full rank, $m \geq n$, and $b \in \mathbb{C}^m$, consider the block 2×2 system of equations

$$\begin{pmatrix} I & A \\ A^* & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix},$$

where I is the $m \times m$ identity. Show that this system has a unique solution $\begin{pmatrix} r \\ x \end{pmatrix}$, and that the vectors r and x are the residual and the solution of the least squares problem "$\|b - Ax\| = \min$".

2. Let $A \in \mathbb{C}^{m\times m}$ be nonsingular. Denote by $A_{1:k,\,1:k}$ the upper-left $k \times k$ block of A. Show that A has an LU factorization (without pivoting, that is) if and only if $A_{1:k,\,1:k}$ is nonsingular for $1 \leq k \leq m$.

* 3. Suppose $A \in \mathbb{C}^{m\times m}$ is banded with *band half-width* p, i.e. $a_{ij} = 0$ whenever $|i - j| > p$.

   (a) If A fulfils the condition of the question 2, what can you say about the sparsity patterns of L and U ?

   (b) If a factorization $PA = LU$ (partially pivoted) is computed, what can you say about the sparsity patterns of L and U ?

4. Show that for Gaussian elimination with partial pivoting applied to any matrix $A \in \mathbb{C}^{m\times m}$, the growth factor

$$\rho(A) = \frac{\max_{i,j} |u_{ij}|}{\max_{i,j} |a_{ij}|},$$

where u_{ij} are the elements of U in $PA = LU$, satisfies $\rho \leq 2^{m-1}$. (Note: here ρ is not the same as the spectral radius.)

* 5. Let $A \in \mathbb{C}^{m\times m}$. Prove that A is unitarily diagonalizable (i.e. $A = Q\Lambda Q^*$, $Q =$ unitary, $\Lambda =$ diagonal) if and only if A is normal (i.e. $AA^* = A^*A$).

6. Let $A \in \mathbb{R}^{m\times m}$ be tridiagonal and symmetric.

   (a) In the QR factorization $A = QR$, which entries of R are nonzero?

   (b) Show that the tridiagonal structure is recovered when the product RQ is formed.