HUT , Institute of mathematics Gripenberg
Mat-1.196 Mathematics of neural networks
Exercise 2
22–29.1.2002

1. Assume that $d \geq 1$ and that $\sigma \in \mathcal{B}^{\infty}_{\mathrm{loc}}(\mathbb{R})$ is such that the closure of the set of discontuities of σ has Lebesgue measure 0 and σ is not (almost everywhere equal to) a polynomial. Show that $S_d(\sigma)$ is dense in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ where $1 \leq p < \infty$.

2. Define the operator Δ_h by $(\Delta_h f)(t) = f(t+h) - f(t)$ where $h > 0$. Show that f is a polynomial of degree at most m if and only if $\Delta_h^{m+1} f = 0$ for all $h > 0$.

3. Define the operator Δ_h by $(\Delta_h f)(t) = f(t+h) - f(t)$ where $h > 0$. Show that if $\varphi * \sigma$ is a polynomial of degree at most m for all infinitely many times differentiable functions that are 0 outside $[-1, 1]$, then $\varphi * (\Delta_h^{m+1}\sigma) = 0$ for all such functions φ.

4. Show that if $\varphi * \sigma$ is a polynomial of degree at most m for all infinitely many times differentiable functions that are 0 outside $[-1, 1]$ then σ is (almost everywhere equal to) a polynomial of degree at most m.
Hint: One can use distribution theory for this or one can choose as the function φ the function $\psi_\lambda(\underline{t}) = \lambda\psi(\lambda\underline{t})$ where $\lambda \geq 1$ and $\psi(t) = 0$ when $|t| \geq 1$, then let $\lambda \to \infty$ and use the exercises above.