HUT , Institute of mathematics Gripenberg

Mat-1.196 Mathematics of neural networks

Exercise 11

26.3–12.4.2002

1. A radial-basis function network calculates the function

$$F(\mathbf{x}) = \sum_{i=1}^{m} c_i \varphi(|\mathbf{x} - \mathbf{x}_i|).$$

Show how such a function can be realized as a multilayer perceptron network.

Solution: Assume that the dimension of $\mathbf{x}$ is d_0. Then we take $d_1 = md_0$ and W_1 to be the matrix with $W_1(i,j) = 1$ if $i + kd_0 = j$ for some $k \geq 0$ and we take τ_1 to be the column vector containing all the vectors $\mathbf{x}_i$. We take $\sigma_1(\underline{t}) = \underline{t}^2$. For the next layer we take $d_2 = m$, $\tau_2 = 0$, $W_2(i,j) = 1$ if $(i-1)d_0 + 1 \leq j \leq id_0$ and 0 otherwise. We take $\sigma_2(\underline{t}) = \varphi(\sqrt{t})$. For the last layer we have $d_3 = 1$, $\tau_3 = 0$ and $W(1,j) = c_j$.

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2. Construct a multilayer perceptron network such that when it is given the input $\mathbf{x}$ its output is $c\mathbf{x}$ where $\frac{1}{c} = \sum_{j=0}^{d_0} \mathbf{x}(j)$ assuming that all components of $\mathbf{x}$ are positive.

Solution: Assume that the dimension of $\mathbf{x}$ is d_0. Choose $d_1 = d_0 + 1$, $\tau_1 = 0$ and take $W_1(i,j) = 1$ if $1 \leq i = j \leq d_0$ and $W_1(d_0+1, j) = 1$ for all j. We take $\sigma_1(\underline{t}) = \ln(\underline{t})$ which is possible since all components are positive. Next we take $d_2 = d_0$, $\tau_2 = 0$, $W_2(i,j) = 1$ if $1 \leq i = j \leq d_0$ and $W_2(i, d_0+1) = -1$. Finally we choose $\sigma_2(\underline{t}) = e^{\underline{t}}$. This network has the desired output.

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3. Suppose one is given a sequence of vectors $\mathbf{x}_n$ such that $\mathbf{x}_{n+k} = \mathbf{x}_n$ for some $k > 1$. Suppose the vectors $\mathbf{y}_n$ are calculated by the formula

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \gamma(\mathbf{x}_n - \mathbf{y}_n),$$

where $0 < \gamma < 1$. What can one say about

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbf{y}_j?$$

Solution: The difference equation can easily be solved and the solution is

$$\mathbf{y}_n = (1-\gamma)^n \mathbf{y}_0 + \sum_{i=0}^{n-1} (1-\gamma)^{n-1-i} \gamma \mathbf{x}_i.$$

Since $0 < \gamma < 1$ we have $\lim_{n\to\infty}(1-\gamma)^n \mathbf{y}_0 = 0$ and hence

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbf{y}_j &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \sum_{i=0}^{j-1} (1-\gamma)^{j-1-i} \gamma \mathbf{x}_i \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} (1-\gamma)^{j-1-i} \gamma \mathbf{x}_i = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (1 - (1-\gamma)^{n-i}) \mathbf{x}_i = \frac{1}{k} \sum_{j=1}^{k} \mathbf{x}_j
\end{aligned}$$

because the sequence is periodic so that $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \mathbf{x}_j = \frac{1}{k}\sum_{j=1}^{k} \mathbf{x}_j$ and because $\sum_{i=0}^{n-1}(1-\gamma)^{n-i} \leq \frac{1-\gamma}{\gamma}$ so that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}(1-\gamma)^{n-i}\mathbf{x}_i = 0.$$
