Mat-1.463 Mathematics 3

Final Examination 16.12.1997 Apiola/Bingham

Please write clearly . . . : see reverse side.

1. Explain all possibilities for the homogeneous equation $Ax = 0$ when A is a 3×5 matrix. Motivate your answer by presenting the different forms that can be obtained using the Gauss elimination. Present these forms after they have been transformed into simple basic forms using appropriate column permutations.

   Use the following notation for the matrix elements:

   - "0" for a number 0,
   - "1" for a number $\neq 0$,
   - "×" for any number (if it makes no difference whether it is 0 or not)

   In each case, state the rank of the matrix and the dimension of the solution space (*i.e.*, the number of free parameters).

2. Compute e^{At} for the matrices a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ c) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

3. Solve the undamped oscillating system

   $$my'' + ky = A_0 \sin \omega t, \quad y(0) = 0, y'(0) = 0$$

   by performing the Laplace transform on the equation and using the convolution theorem in the inverse transform.

   Write $\omega_0 = \sqrt{\frac{k}{m}}$, $A = \frac{A_0}{m}$. Assume that $\omega = \omega_0$ (resonance).

4. Consider a rod of length 10 units, whose material is such that the constant in the heat conduction equation has value $c^2 = 1$, *i.e.* the equation reads $u_t = u_{xx}$.

   In the evening, the rod is cut into two at the middle and one half is put into a container filled with (enough) boiling water at $100°C$. The other half is put into a freezer with temperature $-20°C$. On the next morning, at time $t = 0$, the two halves are put into a heat insulator, connected together, and the two free ends are put into a container with ice water ($0°C$).

   Perform, step by step, the separation of variables for determining the temperature distribution function (substitution into a ready formula is not enough.)

   After how long a time does the middle point of the half taken from the freezer reach the temperature $0°C$? Use here the two first terms of the series.

5. Write the function $\cos z$ in the form $u(x, y)+iv(x, y)$ and show, using the Cauchy–Riemann equations, that $\cos z$ is analytic in the entire complex plane. Derive the differentiation formula.

Laplace transforms

Definition: Given $f(t)$, $Lf = F$, $F(s) = \int_0^\infty f(t)e^{-st}dt$
Notation: $u(t) = H(t)$= unit step function.
$(Lf')(s) = sF(s) - f(0)$, $(Lf'')(s) = s^2F(s) - sf(0) - f'(0)$,

$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$
t^k	$\frac{k!}{s^{k+1}}$	e^{at}	$\frac{1}{s-a}$	$\cosh at$	$\frac{s}{s^2-a^2}$	$\sinh at$	$\frac{a}{s^2-a^2}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

$$(Lf')(s) = sF(s) - f(0), \quad (Lf'')(s) = s^2F(s) - sf(0) - f'(0),$$

$$(Lf^{(n)})(s) = s^nF(s) - s^{(n-1)}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

$$L\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}F(s), \; L(f{*}g) = (Lf)(Lg), (f{*}g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = (g{*}f)(t)$$

$$L\{e^{at}f(t)\} = F(s-a), \quad L\{u(t-a)f(t-a)\} = e^{-as}F(s), \quad L\{\delta(t-a)\} = e^{-as}$$

(continued overleaf)