Mat-1.463 Mathematics 3, autumn 1997

Apiola/Bingham

Mid-term examination 3 / 16.12.1997

Please write clearly on every sheet on separate lines:
1) course title, date
2) student number+letter, IN BLOCK LETTERS surname underlined, all given names
3) study programme (AS, KEM, KON, MAA, MAK, PUU, RYK, TFY, TIK, TUO, SÄH)
4) any former names and study programmes
5) signature.

1. Consider the initial value problem
$$y''' = -6y^4, y(1) = -1, y'(1) = -1, y''(1) = 2.$$
(N.B. the right hand side is "to the power 4")
Transform the equation into a first order system and perform two Euler steps, using step size $h = 0.2$.
Perform in addition one step, from the same starting point ($t = 1$), with Heun's method (see below) using step size $h = 0.4$.
Compare the solutions with the exact solution at the point $t = 1.4$. The exact solution is $y(t) = \frac{1}{t-2}$.
2. Let $f(x) = \begin{cases} x, 0 < x < 1 \\ 1, 1 < x < 2 \end{cases}$. Form the cosine series of f. Draw a graph of the function that the series represents in the interval $[-2, 2]$.
3. Consider a rod of length 10 units, whose material is such that the constant in the heat conduction equation has value $c^2 = 1$, *i.e.* the equation reads $u_t = u_{xx}$.
In the evening, the rod is cut into two at the middle and one half is put into a container filled with (enough) boiling water at $100°C$. The other half is put into a freezer with temperature $-20°C$. On the next morning, at time $t = 0$, the two halves are put into a heat insulator, connected together, and the two free ends are put into a container with ice water ($0°C$).
Perform, step by step, the separation of variables for determining the temperature distribution function (substitution into a ready formula is not enough.)
After how long a time does the middle point of the half taken from the freezer reach the temperature $0°C$? Use here the two first terms of the series.
4. Write the function $\cos z$ in the form $u(x,y)+iv(x,y)$ and show, using the Cauchy–Riemann equations, that $\cos z$ is analytic in the entire complex plane. Derive the differentiation formula.

Formulae

Numerical methods for differential equations

Heun's method $y^*_{n+1} = y_n + hf(t_n, y_n), \quad y(t_0) = y_0$
$y_{n+1} = y_n + \frac{1}{2}h(f(t_n, y_n) + f(t_{n+1}, y^*_{n+1}))$.
GKV $= O(h^2)$

Fourier formulae

General case, f defined in the interval $(-L, L)$

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$
$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$

Half period extensions, f defined in the interval $(0, L)$

Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad a_0 = \frac{1}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$