Computational Methods in Inverse Problems, Mat–1.3626

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1. One of the classical examples in dynamical systems and chaos theory is the iteration of the mapping
$$f : [0,1] \to [0,1], \quad x \mapsto 4x(1-x).$$
Although perfectly deterministic, the function can be used to generate a sample that is chaotic, i.e., it looks like a random sample: Start with a value $x_1 \in [0,1]$, $x_1 \neq 0, 1/2, 1$, and generate the sample $S = \{x_1, x_2, \ldots, x_N\}$ using the algorithm
$$x_{j+1} = f(x_j).$$
Having the sample S, investigate its distribution: calculate the mean and variance, and using the Matlab function `hist`, investigate the behavior of the distribution as the sample size N increases.

2. Consider the Rayleigh distribution: if U and V are two independent random variables, both normally distributed with mean zero and variance σ^2, then the distribution of $X = \sqrt{U^2 + V^2}$ is the Rayleigh distribution,
$$\mathrm{P}\{X < t\} = \frac{1}{\sigma}\int_0^t s \exp\left(-\frac{s^2}{2\sigma^2}\right) ds.$$
Given σ^2, generate a sample $S = \{x_1, x_2, \ldots, x_N\}$ from the Rayleigh distribution. Assume now that you believe that the distribution comes from a *log-normal distribution*, that is, the sample $S' = \{w_1, w_2, \ldots, w_N\}$, $w_j = \log x_j$, is normally distributed. Write a Gaussian parametric distribution and estimate the mean and variance of it. Investigate the viability of the log-normality assumption as follows.

   Denote by w_0 and γ^2 the mean and the variance of your sample S'. If the sample would be normally distributed, then you should have
$$\frac{\text{number of } w_j \text{ less than } t}{N} \approx \frac{1}{\sqrt{2\pi\gamma^2}}\int_{-\infty}^t \exp\left(-\frac{(w'-w_0)^2}{2\gamma^2}\right) dw'.$$
Plot the left and right hand sides and compare. (Hint: Matlab has a useful function `erf`).

3. This example is meant to clarify the meaning of the maximum likelihood estimator. Consider again the Rayleigh distribution,
$$\pi(x) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x \geq 0.$$

Unlike the normal distribution, the maximum of $\pi(x)$ and the center of mass of it do not coincide. Calculate the maximum and the center of mass, and control your result graphically.

Assume that you have just *one* sample point, x_1. Find the maximum likelihood estimator for σ^2 based on this data. Does x_1 coincide with the maximum or the mean?

4. Consider again the additive noise model in one dimensions,

$$Y = AX,$$

where A is a log-normallydistributed multiplicative noise, that is, $\log A \sim \mathcal{N}(\log a_0, \sigma^2)$, and it is assumed to be independent of X. By taking the logarithm, the noise model becomes additive. From this observation, derive the likelihood density $\pi(y \mid x)$.