TRUNCATED SVD AND DECONVOLUTION

Let $f : [0,1] \to \mathbb{R}$ be a signal to be estimated from noisy samples of the convolution integral,

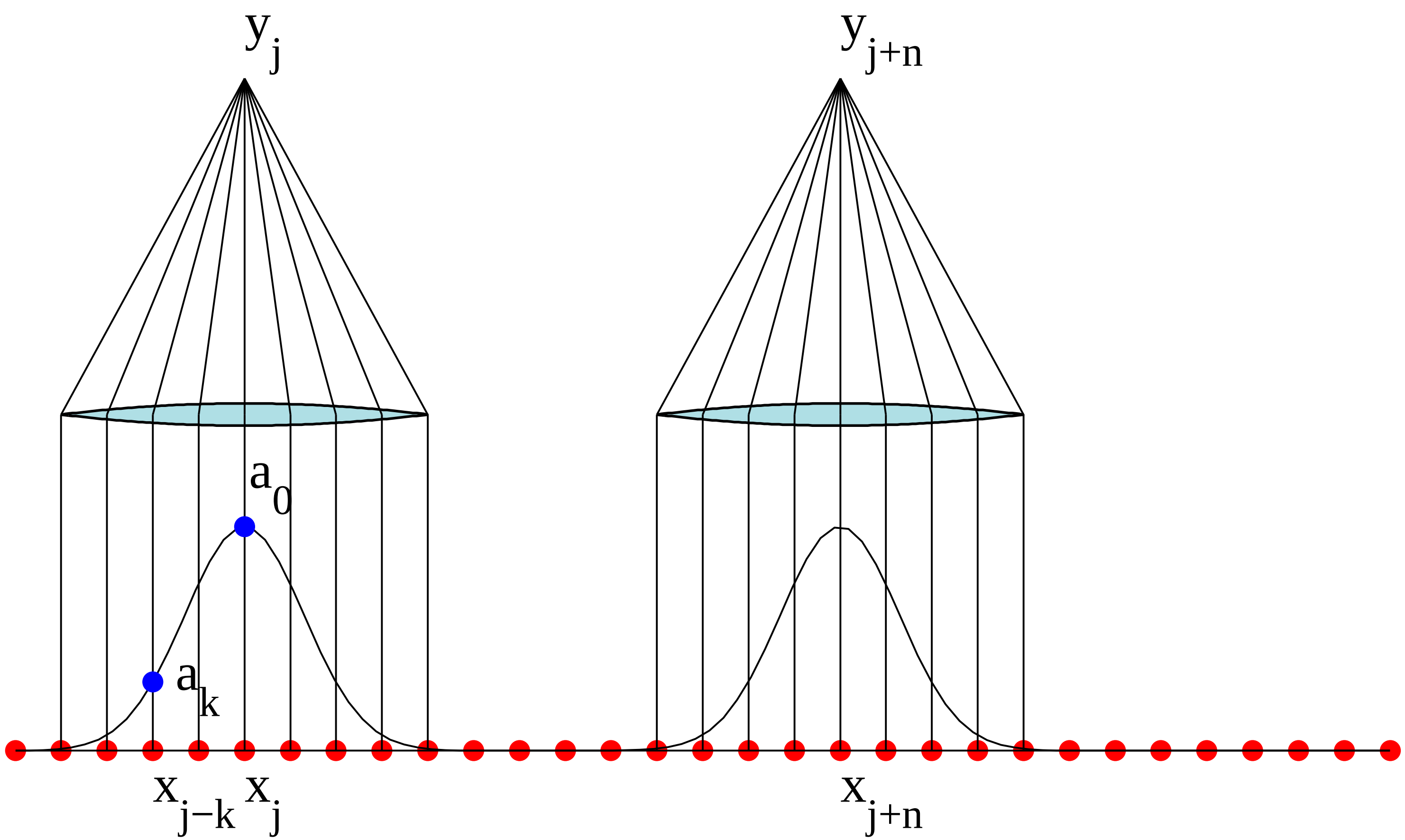
$$g(s) = \int_0^1 a(s-t)f(t)dt + e(s),$$

where a is a known convolution kernel. Discretization:

$$y_j = g(s_j) \approx \frac{1}{N}\sum_k a(s_j - t_k)f(t_k) + e(s_j).$$

Denote $x_k = f(t_k)$, $k = 1, 2, \dots, N$.

Example: Optical blur



Matrix model

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}.$$

Symmetric kernel:

$$a = \begin{bmatrix} a_{-L} \\ a_{-L+1} \\ \vdots \\ a_{L-1} \\ a_L \end{bmatrix} \in \mathbb{R}^{2L+1}.$$

Matrix model

Write the matrix equation

$$y = Ax + e,$$

where $A \in \mathbb{R}^{N\times N}$ is the Toeplitz matrix,

$$A = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{-L} & & \\ a_1 & a_0 & & & \ddots & \\ \vdots & & \ddots & & & a_{-L} \\ a_L & & & \ddots & & \vdots \\ & \ddots & & & a_0 & a_{-1} \\ & & a_L & \cdots & a_1 & a_0 \end{bmatrix}.$$

The parameter L defines the *bandwidth* of the matrix.

Defining the blurring kernel an true signal

Gaussian blurring kernel,

$$a(t) = \frac{1}{\sqrt{2\pi w^2}} \exp\left(-\frac{1}{2w^2}t^2\right).$$

The true signal is a boxcar function,

$$x_j = \begin{cases} 0, & \text{if } j < n_1 \text{ or } j > n_2, \\ 1, & \text{if } n_1 \leq j \leq n_2. \end{cases}$$

In Matlab

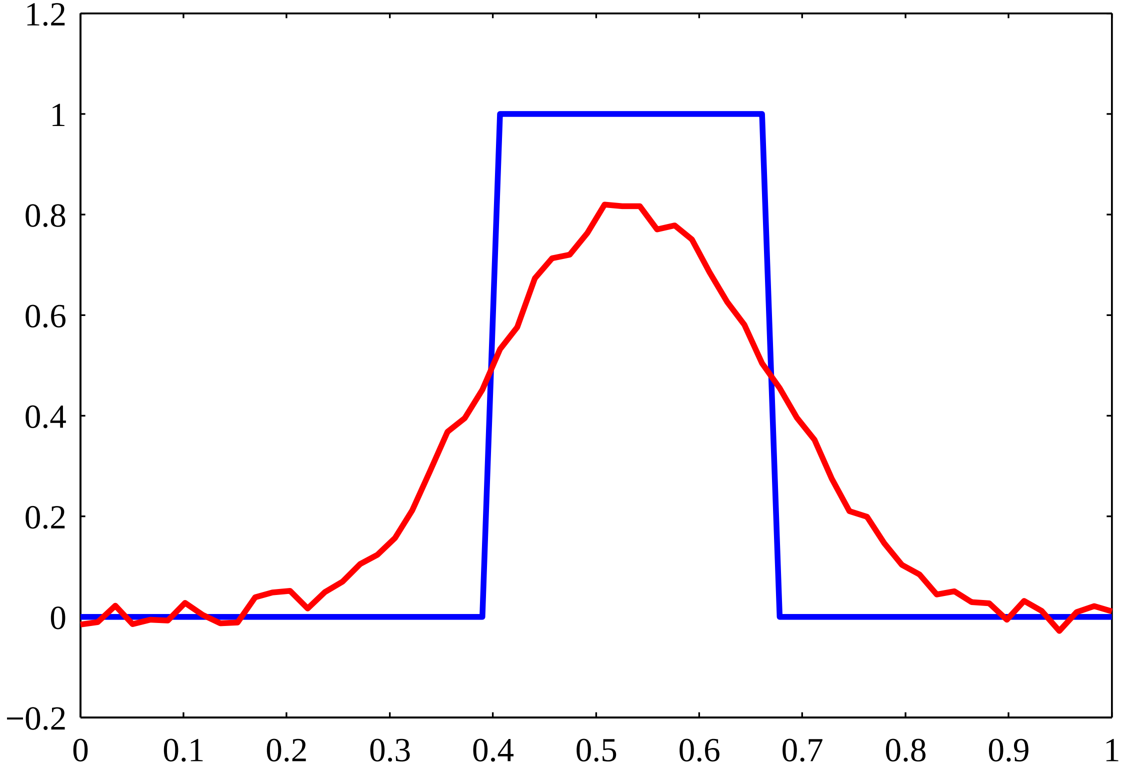
```
N = 60;
t = linspace(0,1,N);
width = 0.1;
a = 1/sqrt(2*pi*width^2)*exp(-(1/(2*width^2))*t.^2);
A = (1/N)*toeplitz(a);

n1 = 25;
n2 = 40;
xtrue = zeros(N,1);
xtrue(n1:n2) = ones(n2-n1+1,1);
```

Adding noise

Gaussian additive noise, the standard deviation (STD) 2% of the maximum of the noiseless signal:

```
b0 = A*xtrue;
noiselevel = 0.02*max(b0);
noise = noiselevel*randn(N,1);
b = b0 + noise;
```

Plot noisy and noiseless signal




```
plot(t,xtrue,'b-','LineWidth',1.2);
hold on
plot(t,b,'r-','LineWidth',1.2);
hold off
```

SVD: plot singular values

$$A = UDV^{\mathrm{T}},$$

where

$$D = \mathrm{diag}(d_1, \ldots, d_n).$$

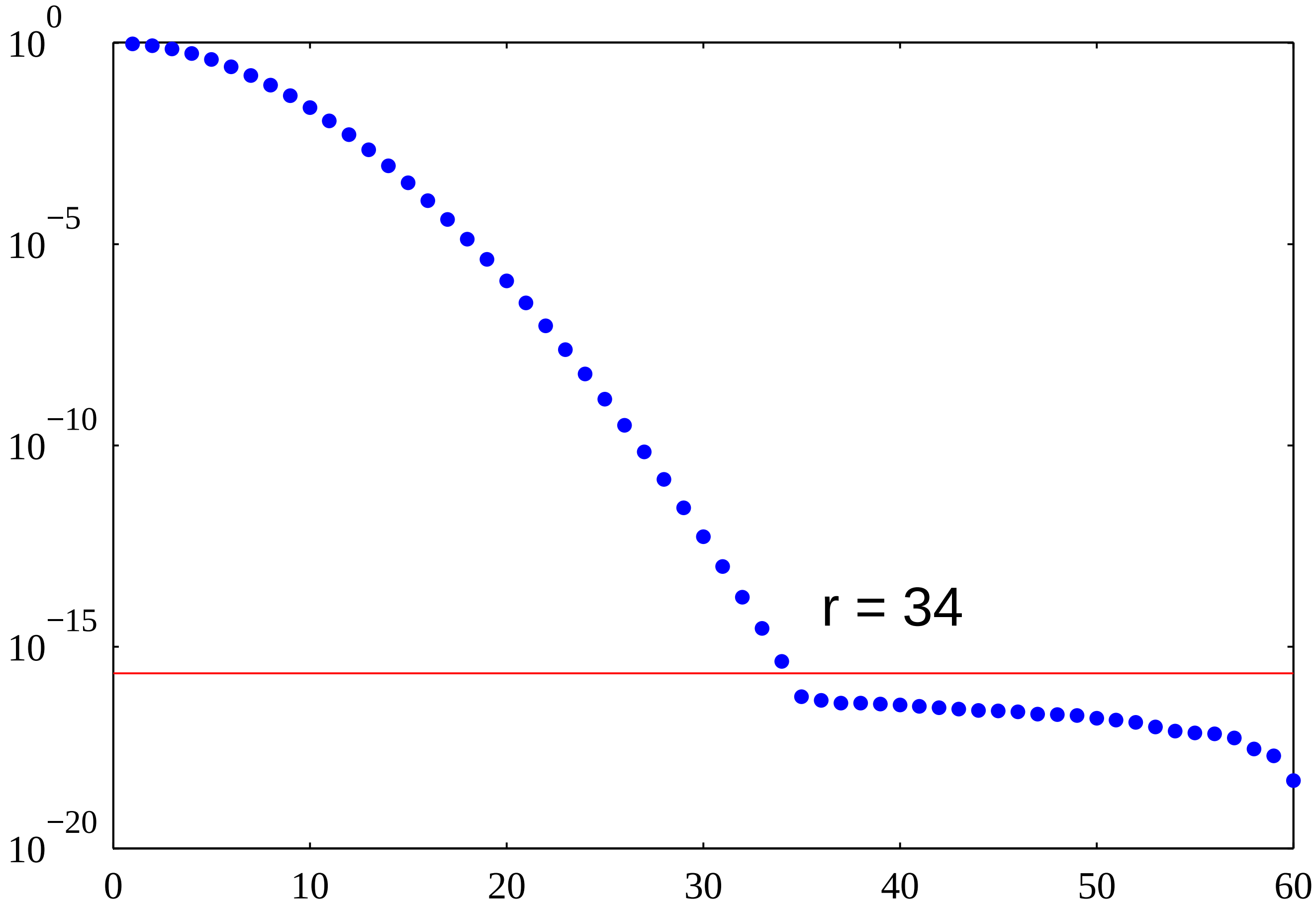
Machine epsilon: smallest non-negative number that the machine recognizes to be non-zero. Below that level, values are cluttered under the roundoff errors.

In Matlab `eps = 2.2204e-016`.

Singular values below `eps` can be treated as zeros.

SVD in Matlab and logarithmic plot

```
[U,D,V] = svd(A);
 d = diag(D);
 r = max(find(d>eps));
 semilogy(d,'b.','MarkerSize',8);
 hold on
 semilogy([0,N],[eps,eps],'r-');
 text(r+2,1e-14,['r = ' ,num2str(r)]);
 hold off
```


r = 34
10^{0}
10^{-5}
10^{-10}
10^{-15}
10^{-20}
0
10
20
30
40
50
60

Calculating TSVD(k)–estimates

$$\widehat{x}^{(k)} = \sum_{j=1}^{k} \frac{1}{d_j}(u_j^{\mathrm{T}} b) v_j.$$

```
Xk = zeros(N,r);
normX = zeros(r,1);
discr = zeros(r,1);
for k = 1:r
    Xk(:,k) = V(:,1:k)*diag(1./d(1:k))*U(:,1:k)'*b;
    normX(k) = norm(Xk(:,k));
    discr(k) = norm(b - A*Xk(:,k));
end
```

Plotting the discrepancy curve

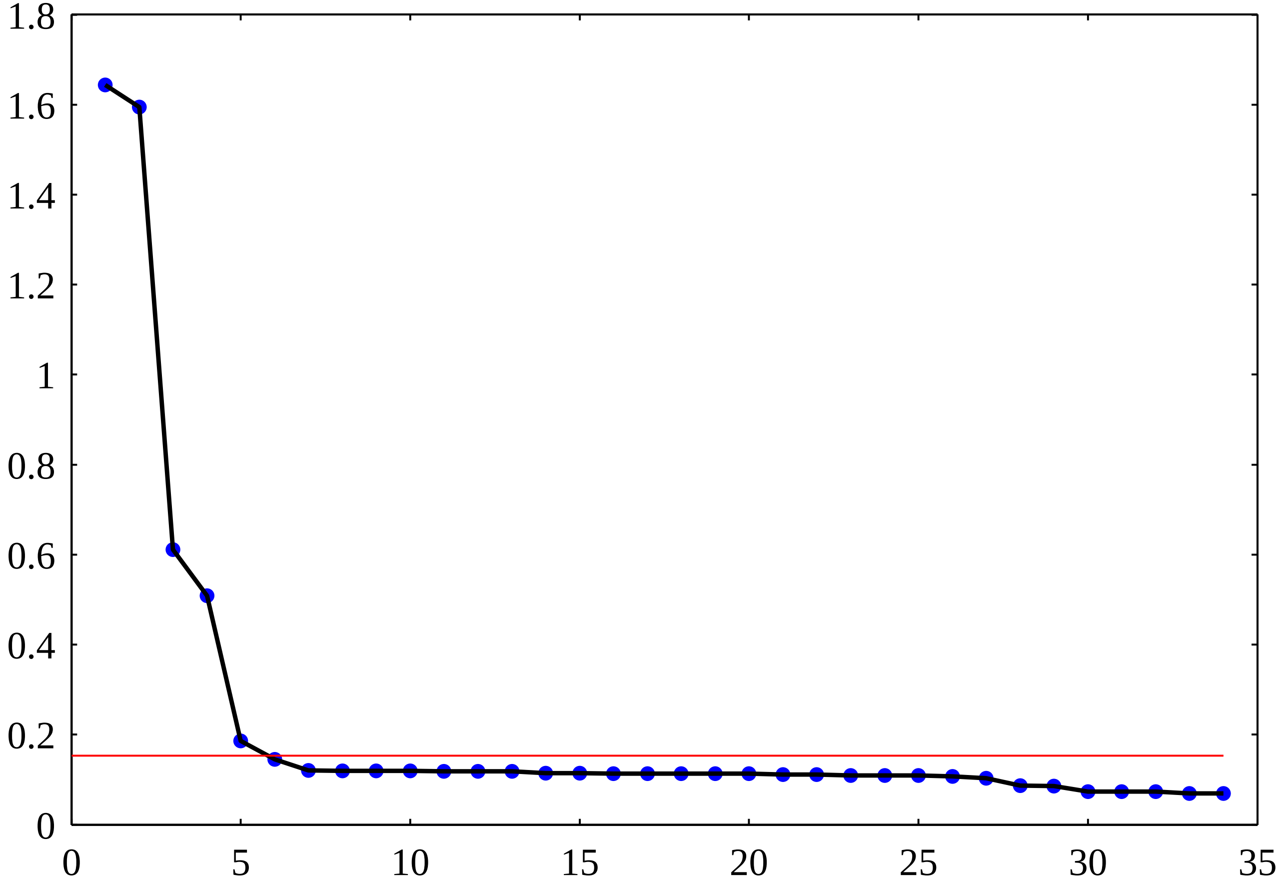
Estimate of the noise level: Here, we set

$$\delta = 1.2\,\|e\|.$$

Notice, that in reality, $\|e\|$ is not known and has to be estimated.

```
plot([1:r],discr,'b.','MarkerSize',8);
hold on
plot([1:r],discr,'k-','LineWidth',0.8)
plot([0,r],[delta,delta],'r-')
hold off
```

PLOTTING THE DISCREPANCY CURVE



Optimal value seems to be $k = 6$ or $k = 7$.

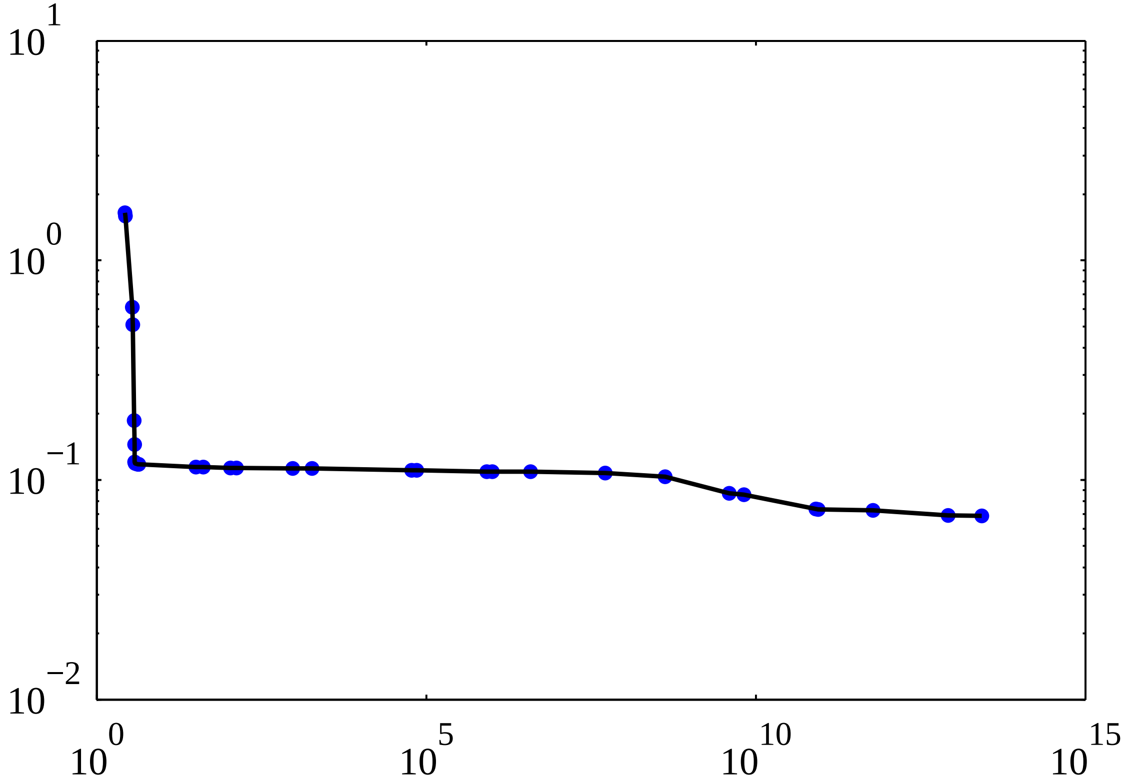
Plotting the L–curve

Plot in log–log scale the points

$$(\|\widehat{x}^{(k)}\|, \|A\widehat{x}^{(k)} - y\|), \quad k = 1, 2, \ldots, r.$$

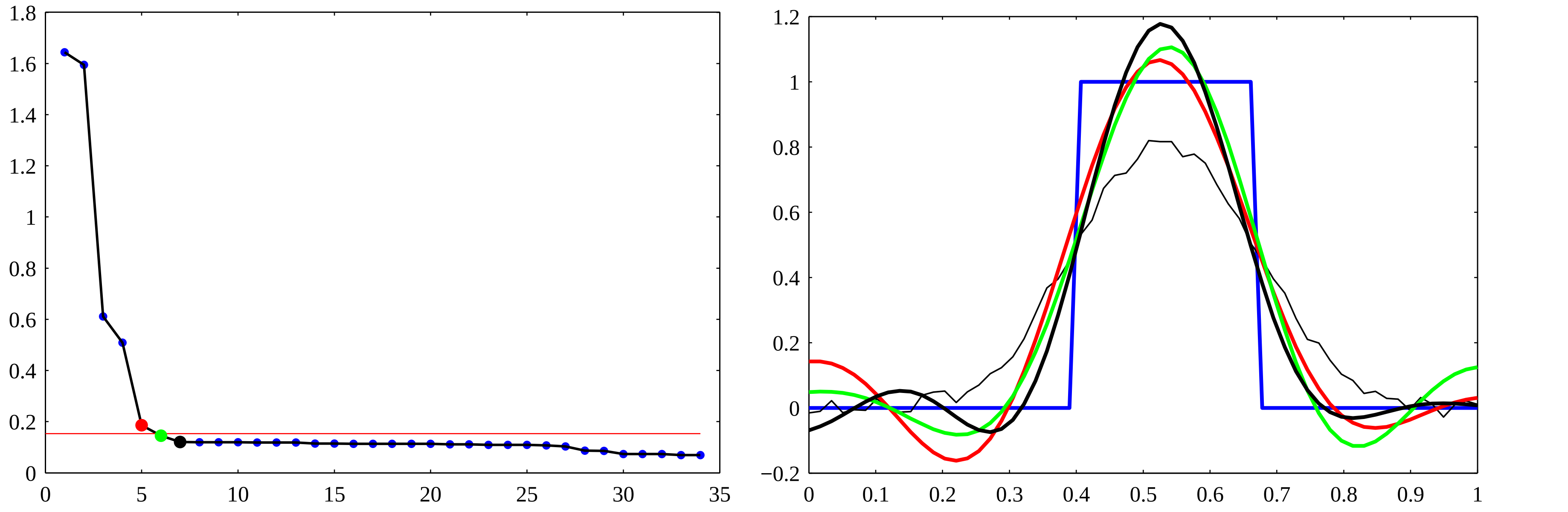
```
loglog(normX,discr,'b.','MarkerSize',8);
hold on
loglog(normX,discr,'k-','LineWidth',0.8)
hold off
```

Plotting the L–curve



Optimal k again around 5–7.

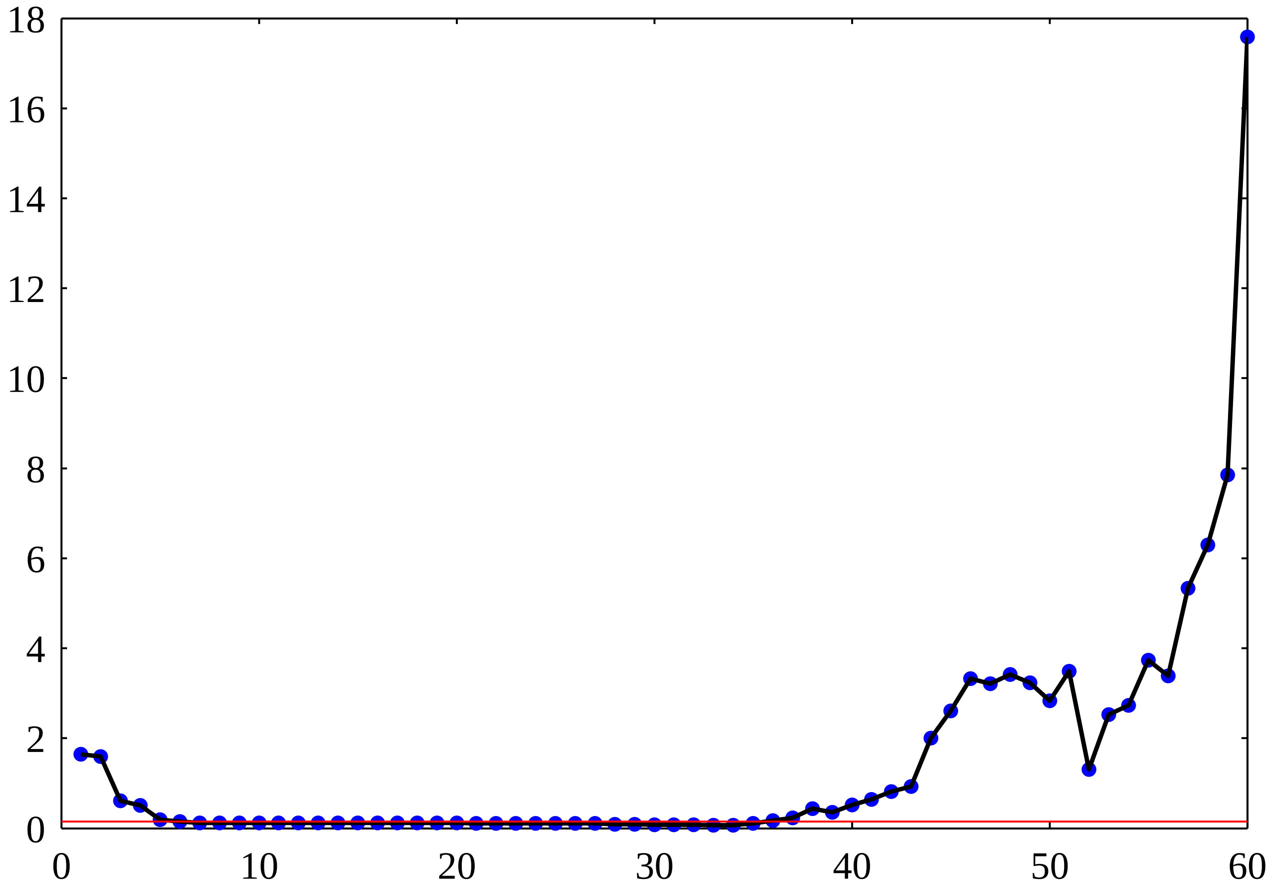
Plotting the solutions



Solutions corresponding to values

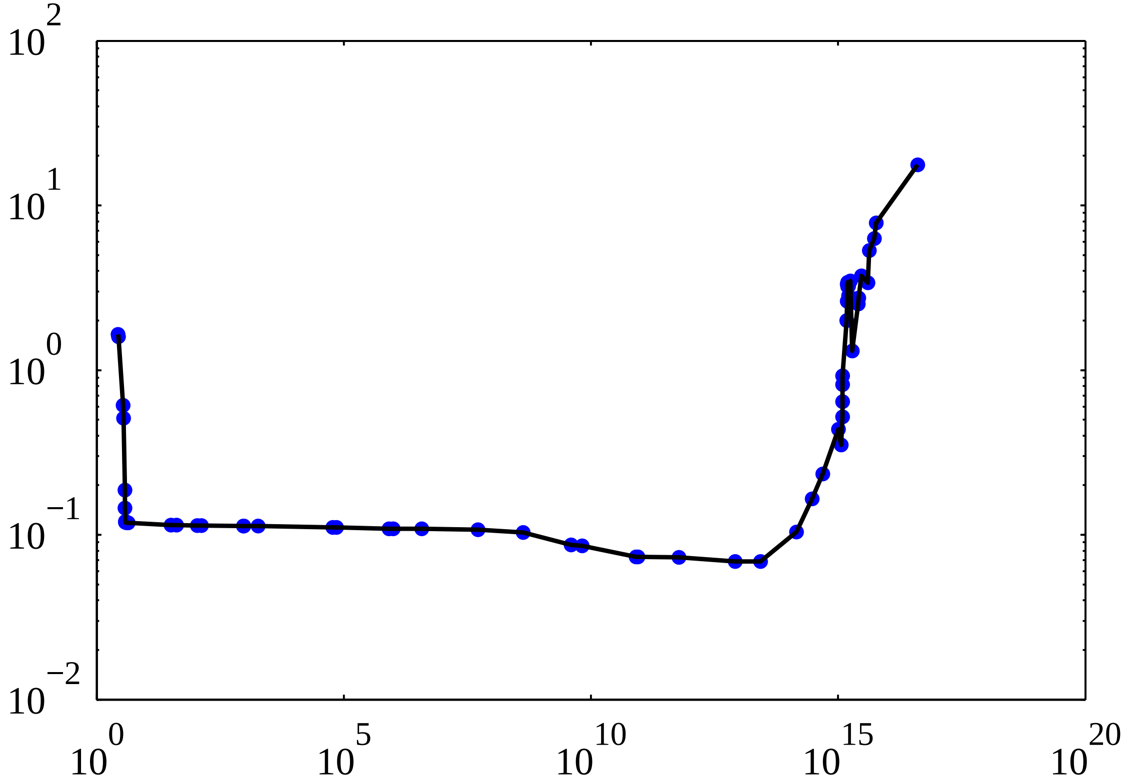
$$k = 5, 6, 7.$$

Is it important to cut off the singular values?



Discrepancy curve with all singular values retained

...Maybe



L–curve with all singular values retained.