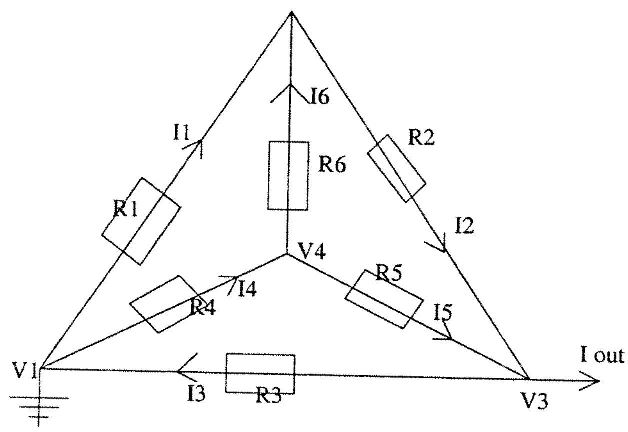
Mat-5.3741 Theory of Elasticity (5 cp) L **Spring 2007**
Stenberg/Juntunen

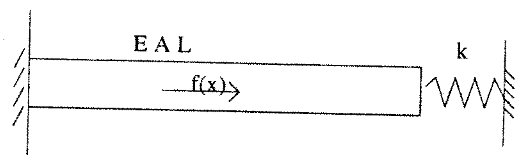
Exercise 2

Problem 1
Consider following network



Write down in matrix form the voltage drops as a function of voltages, the currents as a function of voltage drops, and the conservation of currents. Lastly write down system of equations from which the voltages can be determined.

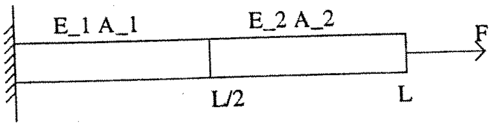
Problem 2 **(home exercise)**
You can choose your this weeks home exercise. Do this one or the MATLAB problem proposed in last weeks exercise paper.
Consider the following rod



where $f(x)$ is a distributed load and k is the spring constant. Write down the energy expression (M), principle of virtual work (V), and boundary value problem (D) for this problem.

Problem 3

Consider the following rod



built of two different materials having different cross sections. As in previous problem, write down the three formulations (M), (V), and (D).

Problem 4

Consider the following rod

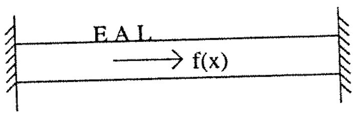


Figure out and/or prove that the solution is given by

$$u(x) = \int_0^L K(x,y)f(y)\,dy,$$

where $K(x,y)$ has been obtained in last weeks problem 2.

Alternate home exercise

Make your own MATLAB program by which simple strusses can be analyzed. Make experiments on systems with and without unique solutions. In the latter case, check the nullspaces of the equilibrium equation and the force compatibility conditions. Compute and print the eigenmodes of some simple structures.

Problem 1

Denote ΔU_i the voltage drop in wire i.

$$\underbrace{\begin{bmatrix} \Delta U_1 \\ \Delta U_2 \\ \Delta U_3 \\ \Delta U_4 \\ \Delta U_5 \\ \Delta U_6 \end{bmatrix}}_{=:\,\Delta U} = \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}}_{=:\,A} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}}_{=:\,V}$$

$AV = \Delta U$ are the voltage drops as a function of voltages.

We know that current in wire i is $I_i = \dfrac{\Delta U_i}{R_i}$, so

$$\underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix}}_{=:\,J} = \underbrace{\begin{bmatrix} 1/R_1 & & & 0 \\ & 1/R_2 & & \\ & & \ddots & \\ 0 & & & 1/R_6 \end{bmatrix}}_{=:\,R^{-1}} \Delta U$$

$R^{-1}\Delta U = J$ are the currents as a function of voltage drops.

The conservation of currents means that the sum of currents coming in and going out has to be zero.

$$\underbrace{\begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix}}_{=:B} J = \underbrace{\begin{bmatrix} 0 \\ 0 \\ I_{out} \\ 0 \end{bmatrix}}_{=:F}$$

$BJ = F$ is the conservation of currents.

We also notice that $B = A^T$!!

Next we want equation from which voltages can be determined.

$$\begin{aligned} & A^T J = F && \mid R^{-1} \Delta U = J \\ \Leftrightarrow \quad & A^T R^{-1} \Delta U = F && \mid AV = \Delta U \\ \Leftrightarrow \quad & A^T R^{-1} A V = F \end{aligned}$$

The "boundary condition" is that $V_1 = 0$. So if we denote $K := A^T R^{-1} A$ and replace the first row of K with $[1\ 0\ 0\ 0]$ we get the voltages from equation $KV = F$.

Problem 2

Let us begin with the boundary value problem. We know that $-(EAu')' = f$ in rod. All we need are the boundary conditions. Obviously $u(0) = 0$. In the other end the displacemet induces force in spring $-ku(L)$ which has to be in balance with the force of rod $EAu'(L)$.

$$(D) \quad \begin{aligned} -(EAu')' &= f \qquad 0<x<L \\ u(0) &= 0 \\ EAu'(L) &= -ku(L) \end{aligned}$$

Next we look for the virtual work expression. We take an arbitrary function

$v \in K = \{\text{admissible displacements}\}$.

and multiply the differential equation and integrate over rod.

$$\int_0^L -(EAu')'v \, dx = \int_0^L fv \, dx .$$

Integration by parts gives

$$\int_0^L EAu'v'\,dx - \Big/_0^L EAu'v = \int_0^L fv\,dx$$

$$\Leftrightarrow \int_0^L EAu'v'\,dx - \underbrace{EAu'}_{=-ku(L)} v\big|_{x=L} + \underbrace{EAu'v\big|_{x=0}}_{=0 \text{ since } v(0)=0} = \int_0^L fv\,dx$$

$$\Leftrightarrow \underbrace{\int_0^L EAu'v'\,dx + k\,u(L)v(L)}_{=: D(u,v)} = \underbrace{\int_0^L fv\,dx}_{=: F(v)}$$

Find $u \in K$ such that

$$D(u,v) = F(v) \quad \forall v \in K \qquad (V)$$

Lastly the energy expression:

$$J(v) := \frac{1}{2} D(v,v) - F(v)$$

$$\min_{v \in K} J(v) \qquad (M)$$

Problem 3

Define
$$E = \begin{cases} E_1 & 0 < x < \frac{L}{2} \\ E_2 & \frac{L}{2} < x < L \end{cases}$$

$$A = \begin{cases} A_1 & 0 < x < \frac{L}{2} \\ A_2 & \frac{L}{2} < x < L \end{cases}$$

With these definitions the internal energy of the rod is

$$\frac{1}{2} \int_0^L EAu'^2 \, dx$$

and external energy is $Fu(L)$.

$$J(v) = \frac{1}{2} \int_0^L EAu'^2 \, dx - Fu(L)$$

$$(M) \quad \min_{v \in K} J(v) \quad , \quad K = \{ v \mid v = 0, \int_0^L EAv'^2 \, dx \neq 0$$

Next we look for the virtual work exp. The minimum u satisfies

$$\frac{d\, J(u+tv)}{dt} \Big|_{t=0} = 0 .$$

$$\frac{dJ(u+tv)}{dt} = \frac{d}{dt}\left[\frac{1}{2}\int_0^L EA\,(u+tv)'^2\,dx - F(u(L)+tv(L))\right]$$

$$= \int_0^L EA\,(u+tv)'v'\,dx - Fv(L)$$

$$= \int_0^L EA\,u'v'\,dx - Fv(L) + t\int_0^L EA\,v'^2\,dx$$

$$\left.\frac{dJ(u+tv)}{dt}\right|_{t=0} = 0$$

$$\Leftrightarrow$$

$$(V)\quad \int_0^L EA\,u'v'\,dx = Fv(L) \quad \forall v\in K.$$

Integrate by parts to get

$$-\int_0^L (EAu')'v\,dx + \underbrace{\Big/_0^L EAu'v}_{= EAu'(L)v(L) - EAu'(0)\underbrace{v(0)}_{=0}} = Fv(L)$$

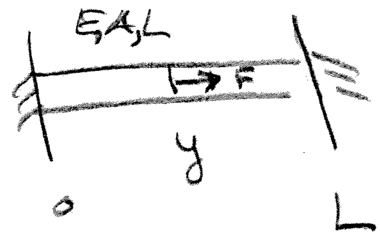
$$\Leftrightarrow -\int_0^L (EAu')'v\,dx + (EAu'(L) - F)v(L) = 0$$

Since v is arbitrary and $u\in K$

$$\left.\begin{aligned} -(EAu')' &= 0 \\ EAu'(L) &= F \\ u(0) &= 0 \end{aligned}\right\}\quad (D).$$

Problem 4

Last time we showed that for problem



there exists $K(x,y)$ s.t.

$$u(x) = K(x,y)\,F.$$

In other words $K(x,y)$ solves equation

$$-(EAu')' = F\delta(x-y) \quad , \; u(0)=u(L)=0$$

$$\Leftrightarrow \quad \mathcal{L}u = F\delta(x-y)$$

$$\Leftrightarrow \quad \mathcal{L}K = \delta(x-y)$$

We are looking for solution to

$$\mathcal{L}u = f(x).$$

We notice that

$$\mathcal{L}u = f(x) = \int_0^L \delta(x-s)\,f(s)\,ds$$

$$= \int_0^L \mathcal{L}K(x,s)\,f(s)\,ds$$

$$= \mathcal{L}\int_0^L K(x,s)\,f(s)\,ds$$

$$\Rightarrow \quad u(x) = \int_0^L K(x,s)\,f(s)\,ds.$$

Problem 5, ex 1

Truss  has unique solution. See next two pages for figures of deformed state and couple of eigenmodes.

Truss  does <u>not</u> have unique solution for all forces.
Mark $K = A^TCA$ where the system to be solved is $A^TCAx = f$.
Compute singularvale decomposition
$K = USV^T$, where $S = \begin{bmatrix}\sigma_1 & \sigma_2 & 0 \\ 0 & \ddots & \sigma_n\end{bmatrix}$
matrix containing the singularvales σ_i and U, V are of full rank and
$UV^T = I$.

$$\Rightarrow \quad V^TKU = S.$$

This means that colums of U related to zero singularvalues belong to $N(K) = \{\text{nullspace of } K\}$

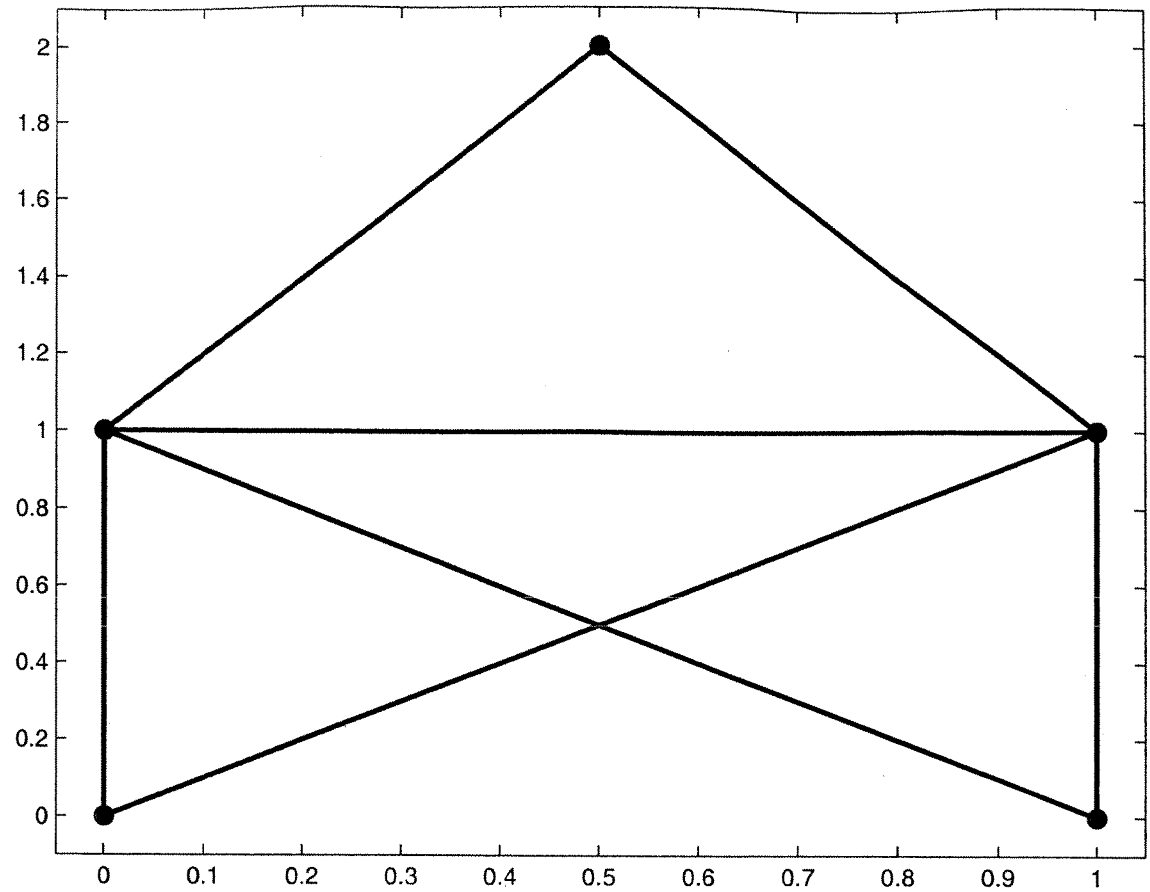
In the first exercise we showed that $N(A) = R(A^T)^\perp$ for all linear operators A. Since $K = K^T$ we get $N(K) = R(K)^\perp$ i.e.

if $x \in N(K)$ then $x \notin R(K)$.

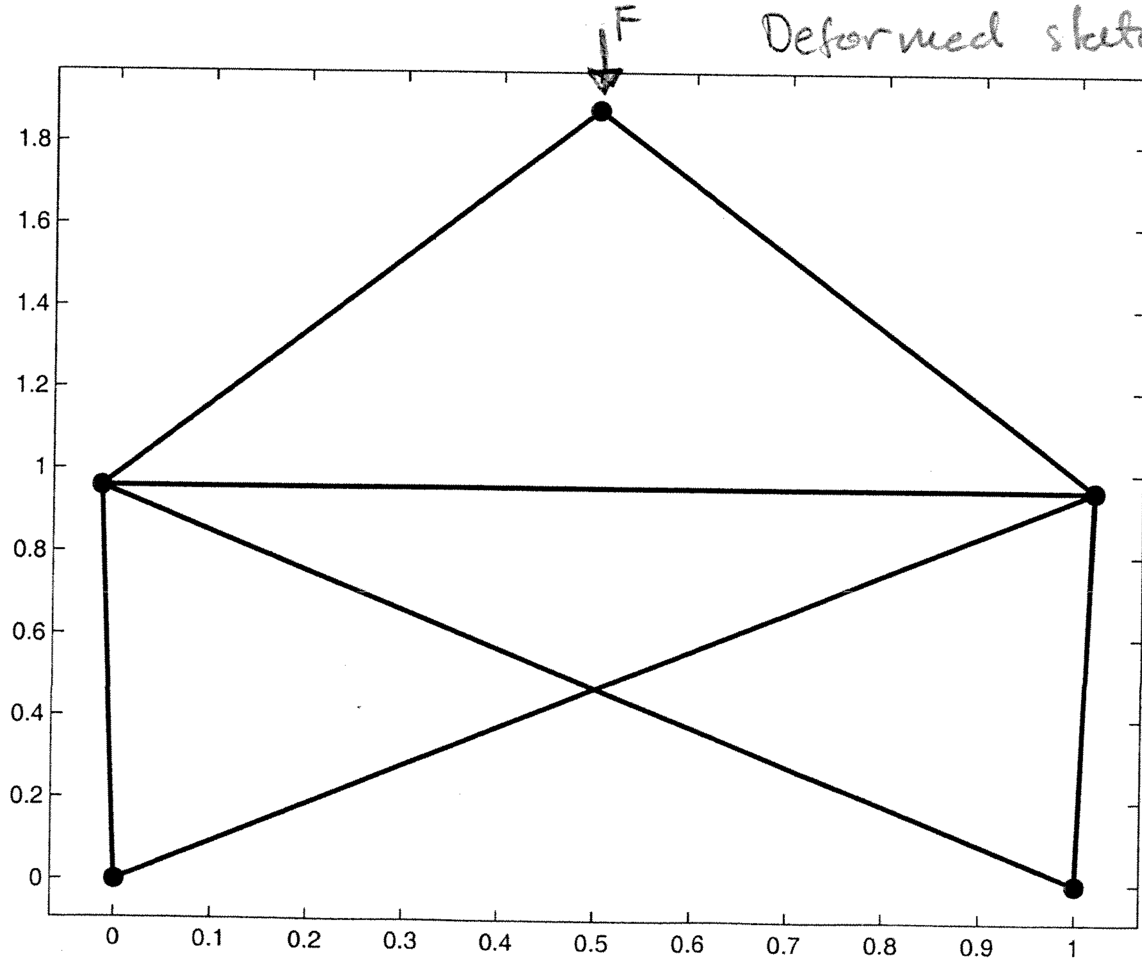
For system $Kx = f$ to have a solution one has to have $f \in R(K)$.

In the following figures there are figures of deformed state of both forces belonging to $R(K)$ and not belonging to $R(K)$.

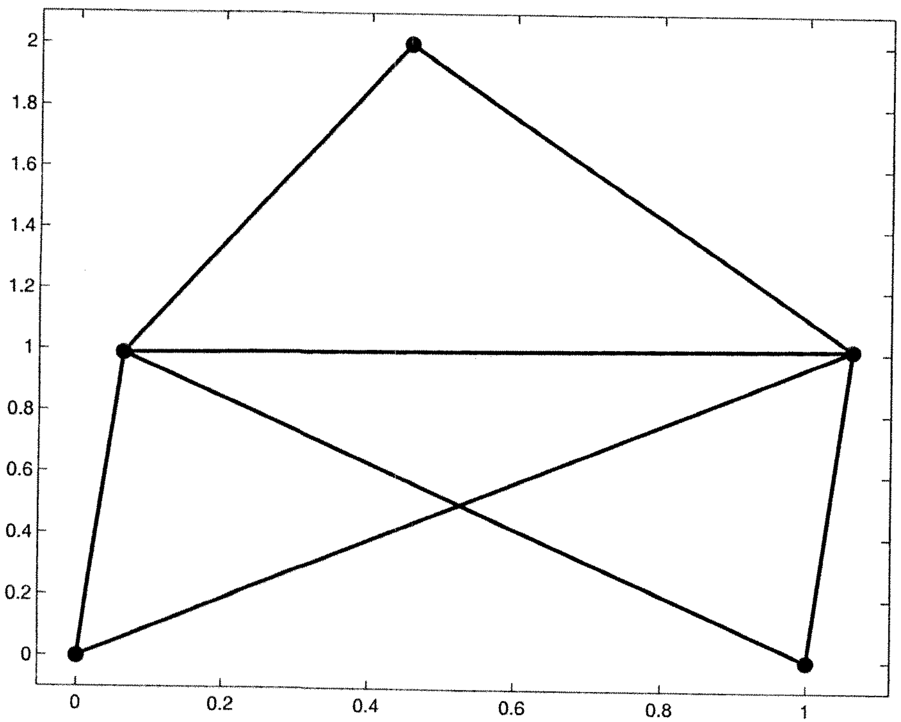
Undeformed truss, unique solution.

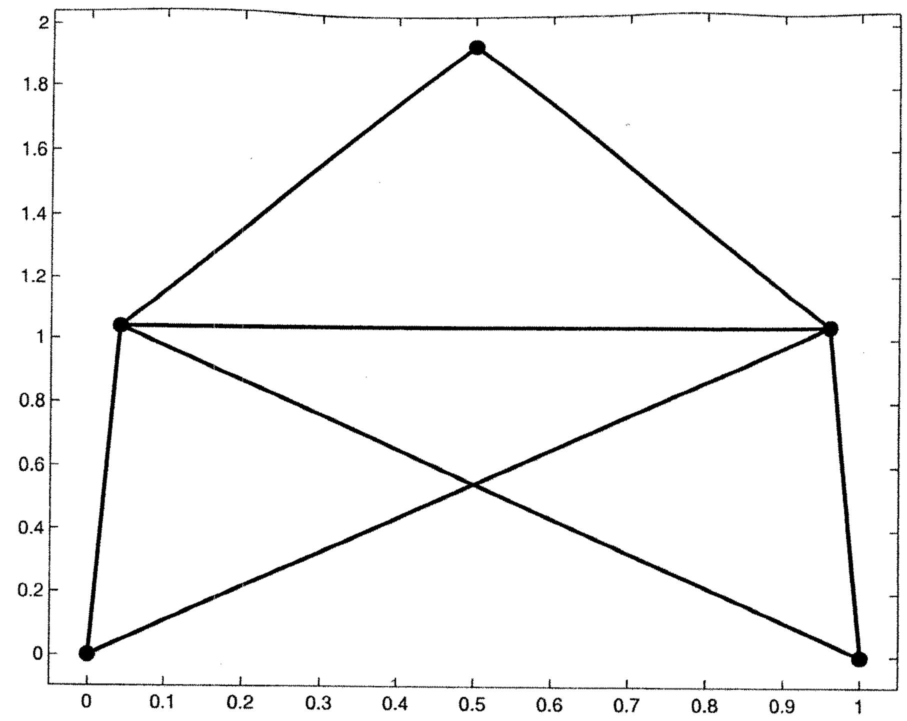


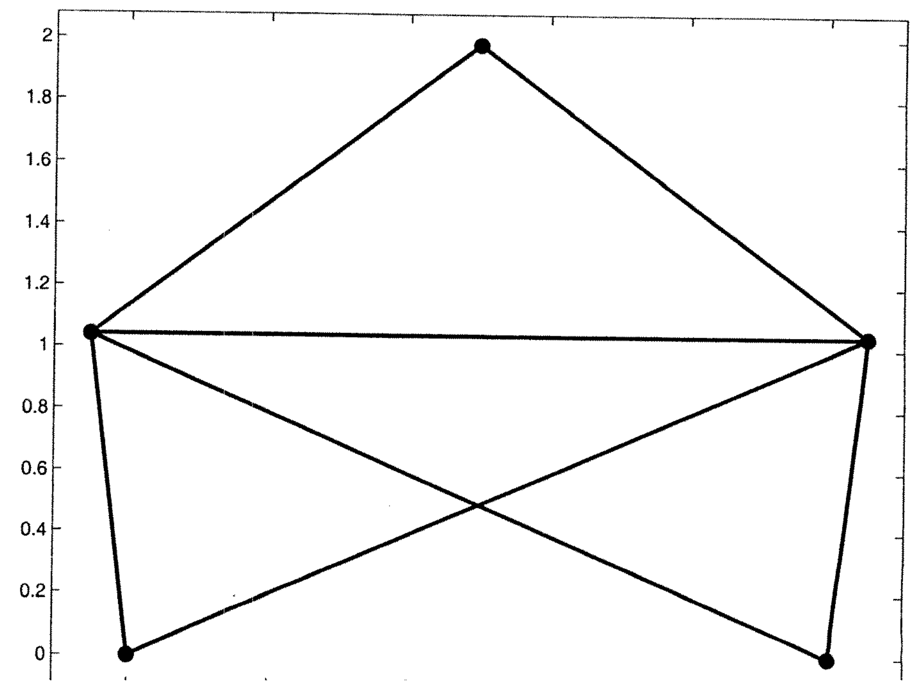
Deformed state.



Couple of eigenmodes of the system with unique solution.

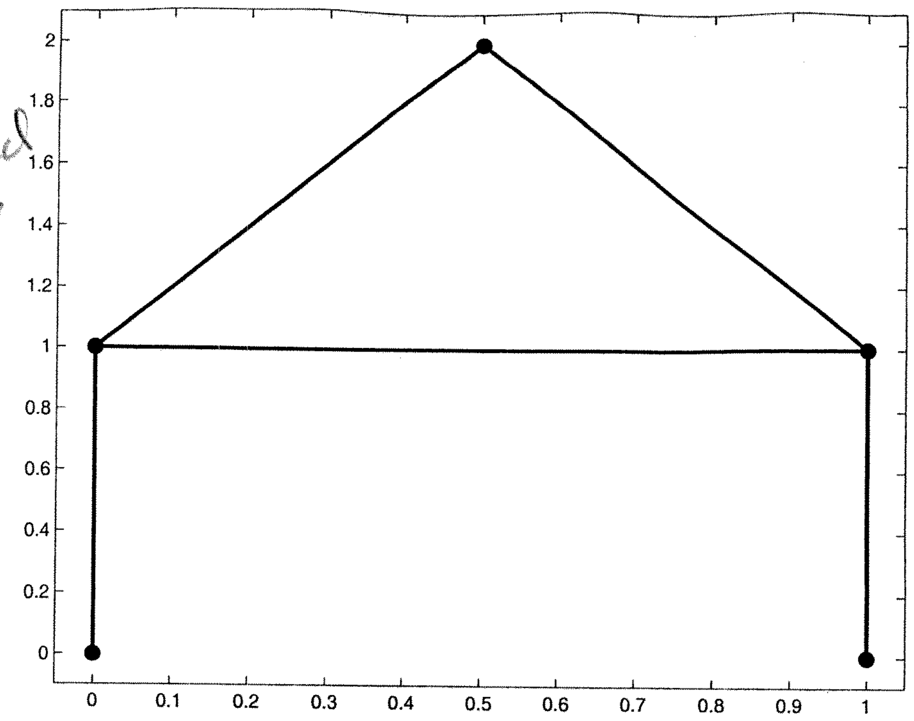




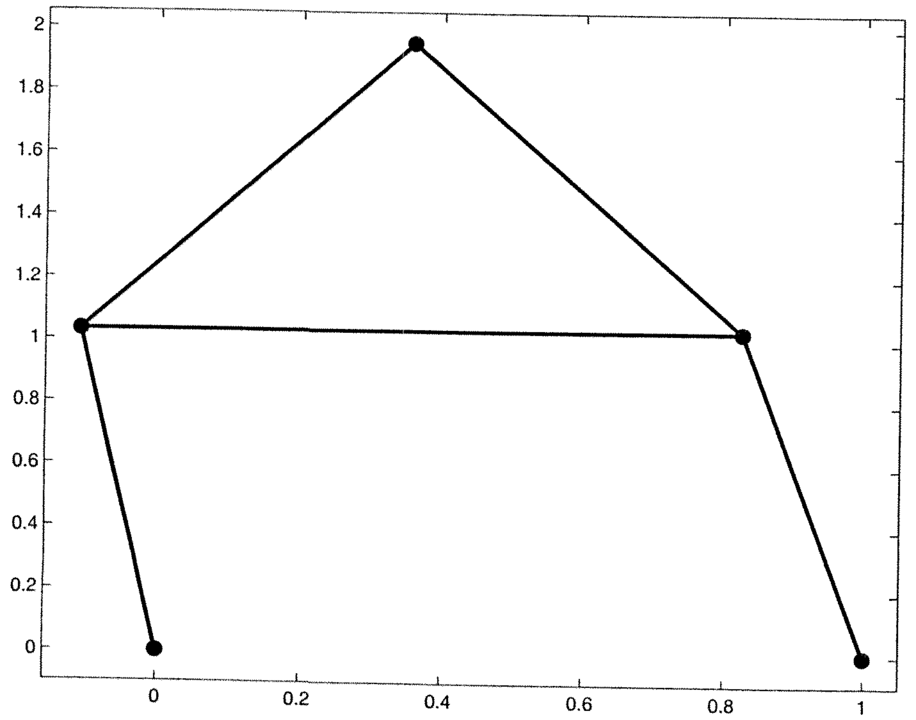


Truss without unique sol.

undeformed



deformed , $f \in R(K)$



Deformed , $f \notin R(K)$