The Truss

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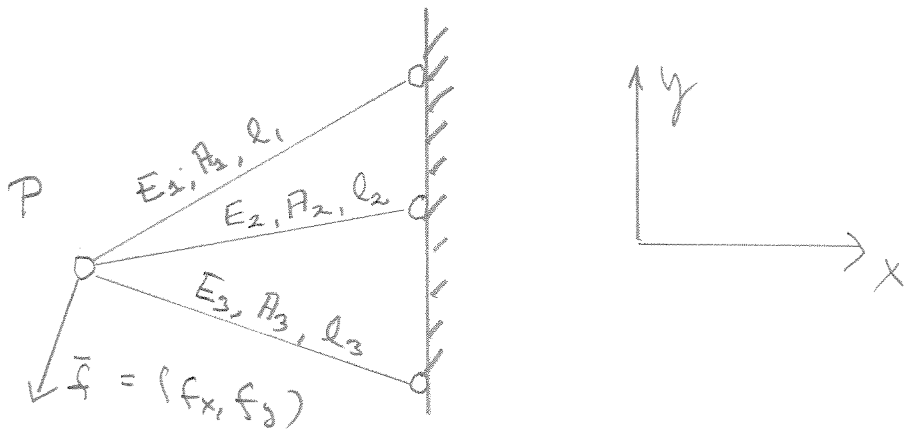
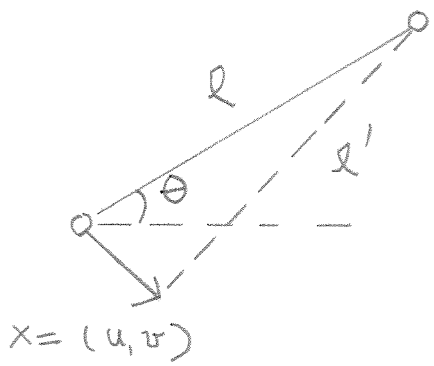
1 Example: A statically indeterminate truss



We have two equilibrium equations, but three unknowns, i.e. the forces in the bars. That is, the structure is statically indeterminate. In order to solve the problem,the elongations of the bars have to be taken into a account.

Let us consider an arbitrary bar and express the elongation using the displacement $x = (u, v)$ of the endpoint:



By Pythagoras we obtain:

$$
\begin{aligned}
(l')^2 &= (l\cos\theta - u)^2 + (l\sin\theta - v)^2 \\
&= l^2\cos^2\theta - 2ul\cos\theta + u^2 + l^2\sin^2\theta - 2vl\sin\theta + v^2 \\
&= l^2 - 2l(u\cos\theta + v\sin\theta) + u^2 + v^2.
\end{aligned}
$$

Naturally, we assume that u and v are "small" which allows us to drop the term $u^2 + v^2$. Hence, we have

$$
\begin{aligned}
(l')^2 &= l^2 - 2l(u\cos\theta + v\sin\theta) \\
&= l^2[1 - \frac{2}{l}(u\cos\theta + v\sin\theta)].
\end{aligned}
$$

Taking square roots, and again using the assumption that u, v are small, we get using the Taylor expansion

$$(1-z)^{\frac{1}{2}} \approx 1 - \frac{1}{2}z \quad 0 < z = \text{"small"},$$

that

$$l' = l[1 - \frac{1}{l}(u\cos\theta + v\sin\theta)].$$

which gives

$$e = l' - l = -u\cos\theta - v\sin\theta.$$

We can now collect the elongation of the three bars into a column vector $(e_1, e_2, e_3)^T = e$ and the displacement as the vector $x = (u, v)^T$.

Then, we write the relation between e and x as a matrix equations:

$$
\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\cos\theta_1 & -\sin\theta_1 \\ -\cos\theta_2 & -\sin\theta_2 \\ -\cos\theta_3 & -\sin\theta_3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}
$$

or more compactly

$$e = Ax\,.$$

The next step is to relate the elongations to the internal forces in the bars. This comes from Hooke's law

$$e_i = \frac{t_i l_i}{E_i A_i}\,,$$

where t_i, is the force in the i-th bar, l_i its length and E_i and A_i are the Young's modulus and cross section, respectively. Writing

$$t_i = c_i\, e_i\,,$$

with

$$c_i = \frac{E_i A_i}{l_i}\,,$$

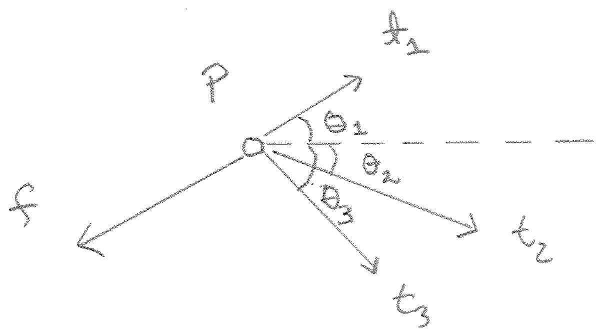
we have

$$t = Ce$$

with

$$t = (t_1, t_2, t_3)^T, \quad C = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}.$$

The third step is to write down the equilibrium equations at the point P.



Equilibrium in the x-direction gives

$$f_x + t_1 \cos\theta_1 + t_2 \cos\theta_2 + t_3 \cos\theta_3 = 0,$$

and in y-direction

$$f_y + t_1 \sin\theta_1 + t_2 \sin\theta_2 + t_3 \sin\theta_3 = 0.$$

Or in matrix form

$$\begin{pmatrix} -\cos\theta_1 & -\cos\theta_2 & -\cos\theta_3 \\ -\sin\theta_1 & -\sin\theta_2 & -\sin\theta_3 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} f_x \\ f_y \end{pmatrix}.$$

That is

$$A^T t = f.$$

The previous matrix A now reappears in *transposed* form.
A summary of what has been done.

A displacement-deformation relationship:

$$e = Ax,$$

A constitutive equation:

$$t = Ce,$$

The equilibrium equations:

$$A^T t = f.$$

Collecting gives a 2×2 system for the deflection $x = (u, v)$:

$$A^T C A x = f.$$

Next, let us make some observations on this system. We recall that a matrix M is

Symmetric if $M = M^T$,
positive definite if $x^T M x > 0$ for all $x \neq 0$,
positive semidefinite if $x^T M x \geq 0$ for all $x \neq 0$.

The constitutive matrix C is clearly symmetric and positive definite, as it is a diagonal matrix with strictly positive entries. Hence, we have

$$(A^T C A)^T = A^T C^T (A^T)^T = A^T C A,$$

i.e. the matrix $A^T C A$ is symmetric. To check the definiteness of it we write

$$C = C^{\frac{1}{2}} \cdot C^{\frac{1}{2}} \quad ,$$

with

$$C^{\frac{1}{2}} = \operatorname{diag}(\sqrt{c_1}, \sqrt{c_2}, \sqrt{c_3}),$$

and we note that

$$(C^{\frac{1}{2}})^T = C^{\frac{1}{2}} .$$

Hence, we have for an arbitrary x

$$\begin{aligned} x^T A^T C A x &= x^T A^T C^{\frac{1}{2}} C^{\frac{1}{2}} A x = x^T A^T (C^{\frac{1}{2}})^T C^{\frac{1}{2}} A x \\ &= (C^{\frac{1}{2}} A x)^T C^{\frac{1}{2}} A x = \|C^{\frac{1}{2}} A x\|^2 \geq 0 , \end{aligned}$$

where $\|\cdot\|$ is the normal Euclidian norm of a vector.
Hence, the coefficient matrix is certainly positive semidefinite.
Further,

$$\|C^{\frac{1}{2}} A x\| = 0 \quad \Leftrightarrow \quad C^{\frac{1}{2}} A x = 0 \quad \Leftrightarrow \quad A x = 0 .$$

The condition $Ax = 0$ for some $x \neq 0$ means a non zero deformation x giving rise to a vanishing elongation in each bar. In the example this is impossible and we have

$$A x = 0 \quad \Leftrightarrow \quad x = 0 .$$

Therefore the coefficient matrix $A^T C A$ *is positive definite* and the problem

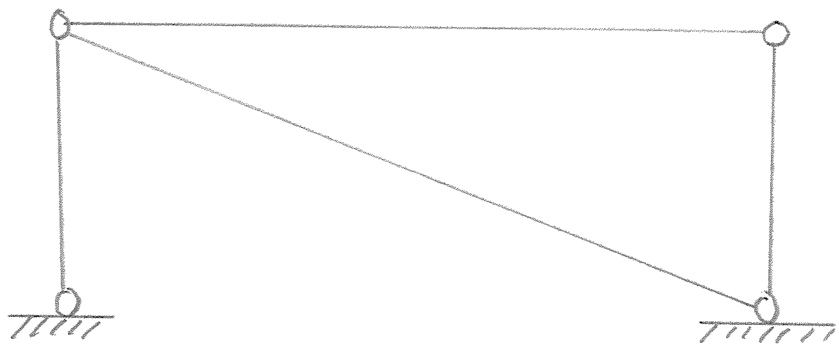
$$A^T C A x = f$$

has a unique solution.

2 The general truss

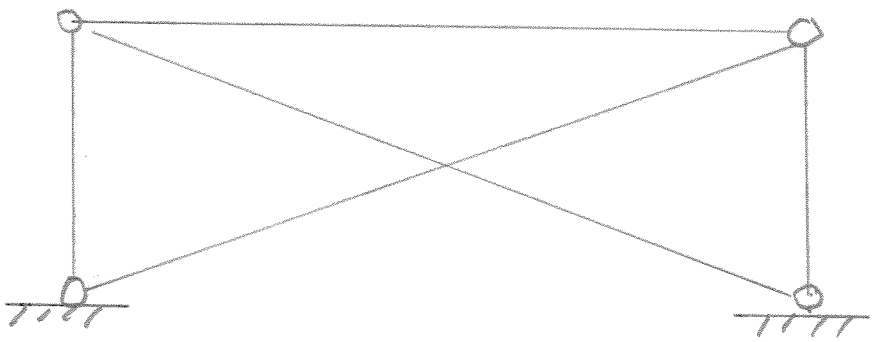
A general truss is either

a) Statically determinate:

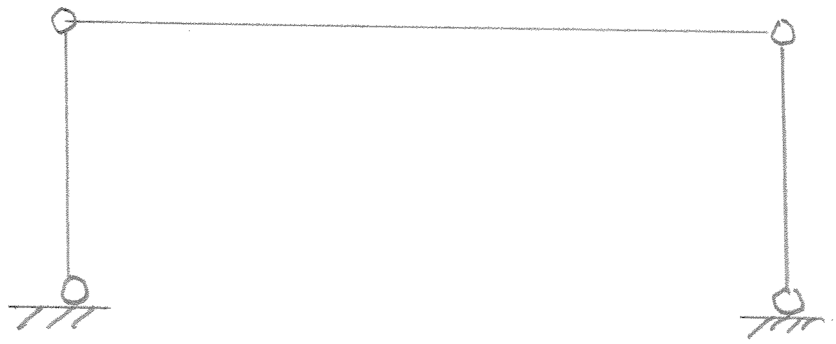


that is, it can be solved by the equilibrium equations alone.

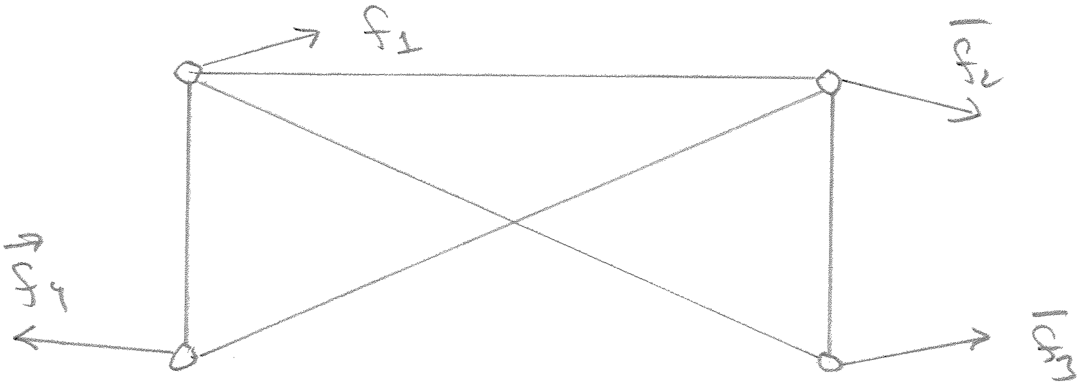
b) Statically indeterminate:



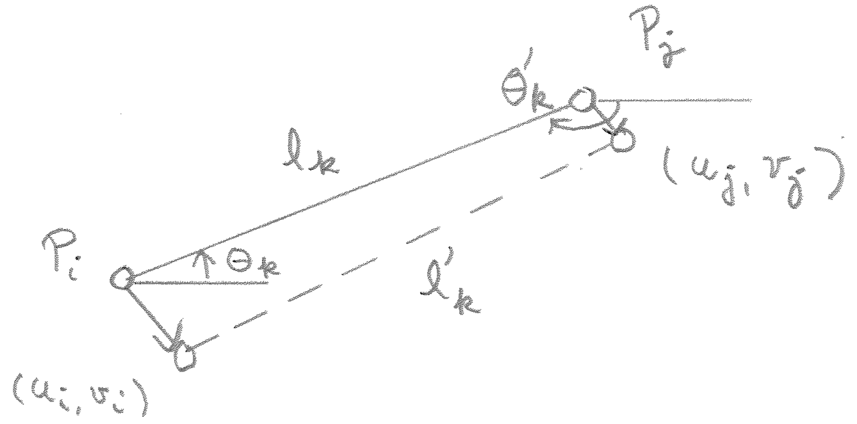
c) Unstable:



The above unstable truss is a *mechanism*. Another form of instability is a *rigid body motion*



The equilibrium equations follows in exactly the same way as for the example. First, we consider an arbitrary bar:



For the elongation e_k we get

$$e_k = -u_i \cos\theta_k - u_j \cos\theta'_k - v_i \sin\theta_k - v_j \sin\theta'_k$$

when the bar k connects the points i and j. This gives a matrix equation

$$e = Ax,$$

where x is the column vector formed by the displacemnts of the joints:

$$x = (u_1, v_1, u_2, v_2, \dots)^T.$$

For a joint we again get two equilibrium equations:

$$\begin{cases} f_x^l + ... + t_k \cos\theta_k + ... = 0. \\ f_y^l + ... + t_k \sin\theta_k + ... = 0. \end{cases}$$

Again, it holds that

$$f = A^T t\,,$$

where t is the vector of forces in the bars and f is the vector of the loads at the joints, i.e.

$$f = (f_x^1, f_y^1, f_x^2, f_y^2, \dots)^T.$$

Finally, we again have that

$$t = Ce\,,$$

with

$$t_i = c_i e_i\,, \quad c_i = \frac{E_i A_i}{l_i}\,.$$

Collecting gives again

$$A^T C A x = f\,.$$

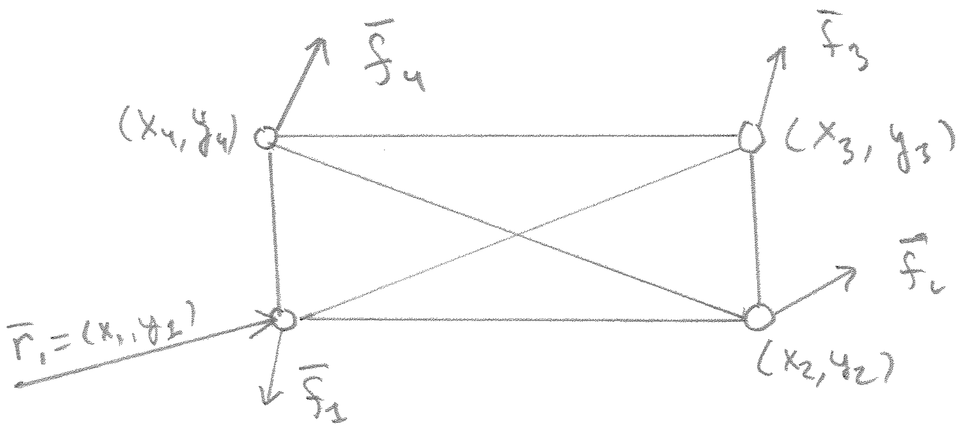
The coefficient matrix is symmetric and positively semidefinite. It is definite when

$$Ax = 0 \quad \Leftrightarrow \quad x = 0\,,$$

i.e. when there are no non-trivial deformations giving zero elongation. For the case of a mechanism of a rigid body motion there is a deformation $x \neq 0$ such that

$$Ax = 0\,.$$

Then the problem does not have a unique solution. Still, there are some combination of forces for which there is a solution. Consider, for instance, a rigid body motion:



For a civil engineer it is clear that the problem can be solved if we have force and moment equilibrium, i.e. if

$$\begin{cases} \sum_{i=1}^{4} \vec{f_i} = \vec{0}, \\ \sum_{i=1}^{4} \vec{r_i} \times \vec{f_i} = \vec{0}. \end{cases}$$

By components, we have

$$\begin{cases} f_x^1 + f_x^2 + f_x^3 + f_x^4 = 0\,, \\ f_y^1 + f_y^2 + f_y^3 + f_y^4 = 0\,, \\ f_x^1 y_1 - f_y^1 x_1 + f_x^2 y_2 - f_y^2 x_2 + \cdots = 0\,. \end{cases}$$

How do we see this mathematically?
Let us go back to a basic result of linear algebra. Let B be a mapping/matrix from $\mathbb{R}^n$ to $\mathbb{R}^m$ i.e. M is a $m \times n$ matrix. We let

$$R(B) = \{\, y \in \mathbb{R}^m \,|\, y = Bx \ \text{ for some } x \in \mathbb{R}^n \}\,,$$
$$N(B) = \{\, x \in \mathbb{R}^n \,|\, Bx = 0\,\}\,,$$

be the *range* and *nullspace*, respectively. Then it holds ("$^\perp$" denotes the orthogonal complement).

Theorem:

$$N(B) = R(B^T)^\perp\,.$$

Exercise: Prove this result.

Now, let us have a look on the equations for the displacement

$$A^T C A x = f.$$

Clearly, it holds

$$R(A^T C A) = R(A^T).$$

Above we proved that $A^T C A x = 0$ is equivalent to $Ax = 0$. Hence, it holds

$$N(A^T C A) = N(A).$$

Now, $N(A)$ consists of the vectors for which $Ax = 0$, that is the rigid body motions spanned by the vectors

$$u_i = 1, \quad v_i = 0, \quad i = 1,2,3,4\,.$$
$$u_i = 0, \quad v_i = 1, \quad i = 1,2,3,4\,.$$

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} -y_i \\ x_i \end{pmatrix}, \quad i = 1,2,3,4\,,$$

i.e. two translations and one rotation. From the theorem we have

$$N(A^T C A) = N(A) = R(A^T C A)^\perp = R(A^T)^\perp\,.$$

A solution to the system exists only if $f \in R(A^TCA) = R(A^T)$ and hence we must have $f \in N(A)^\perp$ that is

$$\sum_{i=1}^{4} f_x^i = 0, \qquad \sum_{i=1}^{4} f_y^i = 0,$$

which means orthogonality against translations, and

$$-\sum_{i=1}^{4} f_x^i\, y_i + \sum_{i=1}^{4} f_y^i\, x_i = 0,$$

i.e. orthogonality against rotations. That is, force and moment equilibrium!

3 The minimum of the potential energy

Let M be a symmetric positive definite matrix. Then the function

$$P(x) = \frac{1}{2} x^T M x - x^T b$$

has a unique minimum at the point where

$$Mx = b.$$

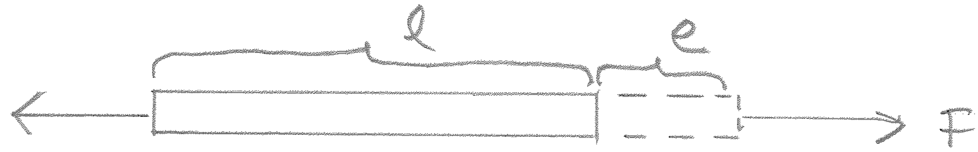
Proof: The Hessian matrix of P is M and the symmetry and positively definiteness guarantees a unique minimum. Suppose x is such that $Mx = b$ and let y be any other vector. We have

$$\begin{aligned} P(y) - P(x) &= \frac{1}{2}\, y^T M y - y^T b - \frac{1}{2}\, x^T M x + x^T b \\ &= \frac{1}{2}\, y^T M y - y^T M x - \frac{1}{2}\, x^T M x + x^T M x \\ &= \frac{1}{2}\, y^T M y - y^T M x + \frac{1}{2}\, x^T M x \\ &= \frac{1}{2}\, (y^T M y - 2 y^T M x + x^T M x) \\ &= \frac{1}{2}\, (y^T M y - y^T M x - x^T M y + x^T M x) \qquad \text{(the symmetry was used here)} \\ &= \frac{1}{2}\, (y - x)^T M (y - x) \geq 0\,. \end{aligned}$$

$\Rightarrow \quad P(y) \geq P(x)$ and equality holds only when $y = x$.

Exercise: Prove this by the traditional method, i.e. from the condition $\frac{\partial P}{\partial x_i} = 0\,,\ i = 1, 2, ..., n\,.$

In mechanics the problem of a truss is usually solved by the principle of the minimum of potential energy (this is what Feng and Shi do). Consider the bar as a spring:



Hooke's law says that $F = ce$ and the elastic energy is

$$\int_0^e F\,de = \int_0^e ce\,de = \frac{1}{2}ce^2\,.$$

Here $c = EA/l$ (as before).
For a truss the sum of the elastic energies is

$$\frac{1}{2}\sum_i c_i\,e_i^2 = \frac{1}{2}e^TCe.$$

Substituting $e = Ax$ gives $\frac{1}{2}\,x^TA^TCAx\,.$
The potential energy of the force is $-x^Tf$ and hence the *total energy* is

$$E(x) = \frac{1}{2}\,x^TA^TCAx - x^Tf$$

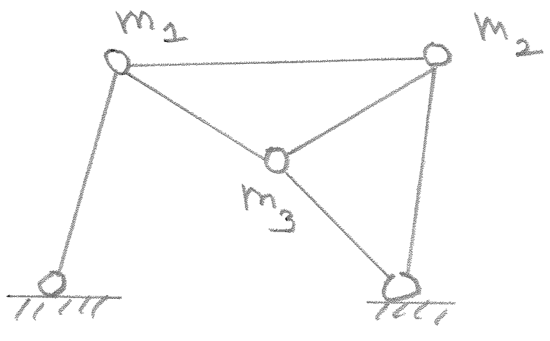
Minimizing E gives

$$A^TCAx = f\,.$$

That is the previous system of equations.

4 Dynamical systems and eigenvalues

Consider next a truss connecting the masses m_i and assume that we can neglect the masses of the bars (not very realistic, but common).



Newtons law for the masses gives

$$M\ddot{x} = -A^Tt,$$

with x being the column vector for the displacemt of the masses, i.e.

$$x = (u_1, v_1, u_2, v_2, \dots)^T,$$

and M is the *mass matrix*

$$M = \mathrm{diag}(m_1, m_1, m_2, m_2, \dots).$$

The "¨" denotes the second derivative with respect to time:

$$\ddot{x} = \frac{d^2x}{dt^2}.$$

Expressing the traction by the displacement, as in the proceeding section, we get

$$M\ddot{x} = -A^TCAx,$$

or with the notation $K = A^TCA$ for the positively definite *stiffness matrix*:

$$M\ddot{x} = -Kx.$$

This system is most easily solved my making the "guess"

$$x(t) = \sin(\omega t)\, y,$$

or

$$x(t) = \cos(\omega t)\, y,$$

with $y \neq 0$ independent of the time t. Taking, e.g. the first choice and substituting into the differential equation gives

$$-\omega^2 \sin(\omega t) My = -\sin(\omega t) Ky.$$

This gives the following equation for the angular velocity ω and the vector y:

$$Ky = \omega^2 My.$$

We note that the second guess gives the same equation. This problem is a *generalized eigenvalue problem* where $\lambda = \omega^2$ is the generalized eigenvalue. Since M and K are symmetric and and K positive definite, this problem has analoguous properties as the (normal) symmetric eigenvalue problem. It holds:

The generalized eigenvalues are real.

The generalized eigenvectors can be chosen to form an "M-orthogonal" basis, i.e.

$$y_i^T M y_j = 0$$

for the the eigenvectors $y_i \neq y_j$.

Furthermore, as K is positively definite, the eigenvalues $\lambda_i = \omega_i^2$ are positive.

Let y_i be the generalized eigenvector for the eigenvalue ω_i^2. We note that

$$x(t) = \sin(\omega_i t)\, y_i,$$

solves the differential equation with the initial conditions

$$x(0) = 0, \quad \dot{x}(0) = \omega_i y_i,$$

whereas

$$x(t) = \cos(\omega_i t)\, y_i,$$

is the solution with initial conditions

$$x(0) = y_i, \quad \dot{x}(0) = 0.$$

Now, let us supplement the differential equations with the arbitrary initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0.$$

As the y_i's form a basis, we can write

$$x_0 = \sum_{i=1}^{n} \alpha_i y_i, \quad \dot{x}_0 = \sum_{i=1}^{n} \beta_i y_i.$$

Since the differential equation is linear, we see that the solution to

$$M\ddot{x} = -Kx, \text{ with } x(0) = x_0 \text{ and } \dot{x}(0) = \dot{x}_0,$$

is given by

$$x(t) = \sum_{i=1}^{n} \alpha_i \cos(\omega_i t) y_i + \sum_{i=1}^{n} \frac{\beta_i}{\omega_i} \sin(\omega_i t) y_i.$$

To find the coefficients α_i and β_i, we use the M-orthogonality:

$$x_0^T M y_k = \sum_{i=1}^{n} \alpha_i y_i^T M y_k = \alpha_k y_k^T M y_k \quad \Rightarrow \quad \alpha_k = \frac{x_0^T M y_k}{y_k^T M y_k}.$$

Similarly, we get

$$\beta_k = \frac{\dot{x}_0^T M y_k}{y_k^T M y_k}.$$

We have obtained an explicit solution to the problem. Let us next show that the solution is *unique*. This we do by using an *energy argument*. We recall that the elastic energy of the system is

$$\frac{1}{2} x^T K x.$$

The kinetic energy is given by

$$\frac{1}{2} \dot{x}^T M \dot{x},$$

and hence the total energy is

$$E = \frac{1}{2} \dot{x}^T M \dot{x} + \frac{1}{2} x^T K x.$$

It now holds.

Theorem. *The energy is conserved.*

Proof. Using the chain rule, the symmetry of M and K, and the differential equation we get

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}\dot{x}^T M\dot{x} + \frac{1}{2}x^T Kx\right) = \dot{x}^T M\ddot{x} + \dot{x}^T Kx = \dot{x}^T(M\ddot{x} + Kx) = 0.$$

To prove the uniqueness, we have to show that the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ implies that $x(t) = 0$ for all $t > 0$ (by linearity). The total energy vanish when $t = 0$ and as it is conserves it vanishes for all $t > 0$. If $x(t) \neq 0$ the positive definiteness of M and K gives

$$0 = \frac{1}{2}\dot{x}^T M\dot{x} + \frac{1}{2}x^T Kx > 0$$

which is a contradiction. Hence, $x(t) \equiv 0$ and the solution is unique.