ryhmät ja monistot

Harj. 1

1) Olkoon $\mathbb{K}$ joukko ja $\begin{cases} + : \mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{K} \\ \cdot : \mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{K} \end{cases}$ kuvauksia s.e.

(1) $a \cdot (b+c) = a \cdot b + a \cdot c$ $\quad a, b, c \in \mathbb{K}$

$(b+c) \cdot a = b \cdot a + c \cdot a$

(2) $(\mathbb{K}, +)$ on Abelin ryhmä

(3) $(\mathbb{K} \setminus \{0\}, \cdot)$ on ryhmä.

Tällöin $\mathbb{K} = (\mathbb{K}, \cdot, +)$ on vinoryhmä.

Määritellään $\mathbb{R}^4$:ään kuvauksia

$$\cdot : \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

$$((a,b,c,d),(x,y,z,w)) \longmapsto (ax - by - cz - dw,\ ay + bx + cw - dz,\ az - bw + cx + dy,\ aw + bz - cy + dx), \quad a, \dots, w \in \mathbb{R}$$

$$+ : \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4, \quad u, v \in \mathbb{R}^4.$$

$$(u, v) \longmapsto u + v$$

tavallinen yhteenlasku $\mathbb{R}^4$:ssa

<u>Merkitään</u> • $\mathbb{H} = (\mathbb{R}^4, +, \cdot)$

• $\mathbb{R}^4$ kantavektorit merkitään

$1 = (1,0,0,0)$, $i = (0,1,0,0)$, $j = (0,0,1,0)$, $k = (0,0,0,1)$

• Jos $\lambda \in \mathbb{R}$, merkitään $\lambda \cdot 1 = \lambda \in \mathbb{H}$.

Näin tulkitaan $\mathbb{R} \subset \mathbb{H}$

Määritellään lisäksi kuvauksia $\bar{\ } : \mathbb{H} \longrightarrow \mathbb{H}$

$$(a,b,c,d) \longmapsto (a,-b,-c,-d)$$

$$|\cdot| : \mathbb{H} \longrightarrow \mathbb{R} \subset \mathbb{H}$$

$$u \longmapsto \sqrt{u \cdot \bar{u}}$$

<u>Huom:</u> $\lambda u = u \lambda$ jos $u \in \mathbb{H}$, $\lambda \in \mathbb{R} \subset \mathbb{H}$.

Huom: • Jos $u = (a,b,c,d) \in \mathbb{H}$, niin $u \cdot \bar{u} = a^2 + b^2 + c^2 + d^2$.

• $|u| = 0 \iff u = 0 =$ id. alkio $+$:n suhteen.

• $\bar{\bar{u}} = u$ joten $u \cdot \bar{u} = \bar{u} \cdot u$.

Väite: $\mathbb{H}$ on vinokunta

(1) $\bullet$-kuvaus voidaan kirjoittaa muodossa

$$u \cdot v = (u \cdot A \cdot v,\ u \cdot B \cdot v,\ u \cdot C \cdot v,\ u \cdot D \cdot v),\ u,v \in \mathbb{R}^4$$

joillakin 4×4 matriiseilla A, B, C, D. $\leadsto$ (1) pätee

(2) $(\mathbb{R}^4, +)$ on Abelin ryhmä sillä $(\mathbb{R}, +)$ on.

(3) $(\mathbb{R}^4 \setminus \{0\}, \bullet)$ on ryhmä

Näytetään ensin, että löytyy kuvaus

$$\varphi_1 : \mathbb{R}^4 \longrightarrow M_{2\times 2}(\mathbb{C})$$

s.e.

$$\begin{cases} \varphi_1(u \bullet v) = \varphi_1(u) \bullet \varphi_1(v), & u,v \in \mathbb{H} \\ \varphi_1 : \mathbb{R}^4 \longrightarrow \varphi_1(\mathbb{R}^4) & \text{bijektio} \end{cases}$$

Tod: Jos $(a,b,c,d) \in \mathbb{R}^4$ määritellään

$$\varphi_1(a,b,c,d) = \begin{pmatrix} s & r \\ -\bar{r} & \bar{s} \end{pmatrix}, \quad s = a + ib, \quad r = c + id$$

i: $\in \mathbb{C}$ ja ei ole sama kuin $i \in \mathbb{H}$!

Alkioille $\underbrace{(a,b,c,d)}_{=u}$, $\underbrace{(x,y,z,w)}_{=v} \in \mathbb{R}^4$ saadaan

$$\varphi_1(u \cdot v) = \begin{pmatrix} s & r \\ -\bar{r} & \bar{s} \end{pmatrix} \qquad \begin{aligned} s &= (ax - by - cz - dw) + i(ay + bx + cw - dz) \\ r &= (az - bw + cx + dy) + i(aw + bz - cy + dx) \end{aligned}$$

$$\varphi_1(u) \cdot \varphi_1(v) = \begin{pmatrix} u_1 & u_2 \\ -\bar{u}_2 & \bar{u}_1 \end{pmatrix} \cdot \begin{pmatrix} v_1 & v_2 \\ -\bar{v}_2 & \bar{v}_1 \end{pmatrix} \qquad \begin{aligned} u_1 &= a + ib \\ u_2 &= c + id \\ v_1 &= x + iy \\ v_2 &= z + iw \end{aligned}$$

$$= \begin{pmatrix} u_1 v_1 - u_2 \bar{v}_2 & u_1 v_2 + u_2 \bar{v}_1 \\ -\overline{(u_1 v_2 + u_2 \bar{v}_1)} & \overline{(u_1 v_1 - u_2 v_2)} \end{pmatrix}$$

$$u_1 v_1 - u_2 \bar{v}_2 = \cdots = s, \qquad u_1 v_2 + u_2 \bar{v}_1 = \cdots = r \qquad \square$$

a) $(1,0,0,0)$ on $\mathbb{H}\setminus\{0\}$:ssa id. alkio;

b) $u,v,w \in \mathbb{H}\setminus\{0\} \Rightarrow u\cdot(v\cdot w) = \varphi_1^{-1}\varphi_1(u\circ(v\circ w)) = \dots = (u\cdot v)\cdot w$

c) Jos $u \in \mathbb{H}\setminus\{0\}$ alkialle $w = |u|^{-2}\cdot\bar{u} \in \mathbb{H}\setminus\{0\}$ pätee

$$w\cdot u = |u|^{-2}\cdot\bar{u}\cdot u = |u|^{-2}\cdot|u|^2 = 1 = (1,0,0,0)$$

$$u\cdot w = u\cdot|u|^{-2}\cdot\bar{u} = |u|^{-2}\cdot u\cdot\bar{u} = (1,0,0,0) \qquad \square$$

1.2 Kvaternionille $q = a + bi + cj + dk$ määritellään

$$\mathrm{Re}\, q = a, \qquad \mathrm{Im}\, q = bi + cj + dk.$$

<u>Väite:</u>

$$\langle u,v\rangle_{\mathbb{R}^3} = -\mathrm{Re}\{V(u)\cdot V(v)\} \qquad u,v\in\mathbb{R}^3$$

$$V(u\times v) = \mathrm{Im}\{V(u)\cdot V(v)\}$$

eli

$$V(u)\cdot V(v) = -\langle u,v\rangle_{\mathbb{R}^3} + V(u\times v)$$

kun $V : \mathbb{R}^3 \longrightarrow \mathbb{H}$

$$(a,b,c) \longmapsto ai + bj + ck = (0,a,b,c)$$

<u>Tod.</u> Olk. $u = (b,c,d)$ $v = (y,z,w) \in \mathbb{R}^3$

$$V(u)\cdot V(v) = (-by - cz - dw,\ cw - dz,\ -(bw - dy),\ bz - cy)$$

$$= -\langle u,v\rangle_{\mathbb{R}^3} + \begin{vmatrix} i & j & k \\ b & c & d \\ y & z & w \end{vmatrix} = -\langle u,v\rangle + V(u\times v).$$

determinantti kehitetään ensimmäisen rivin suhteen $\square$

(1.3) Jos $q_1, q_2 \in \mathbb{H} \setminus \mathbb{R}$ niin

$$q_1 \cdot q_2 = q_2 \cdot q_1 \iff \operatorname{Im} q_1 = \lambda \cdot \operatorname{Im} q_2 \text{ jollakin } \lambda \in \mathbb{R}.$$

(Jos $q_2 = 1$, $q_1 = i$, niin $q_1 q_2 = q_2 q_1$ mutta $i \neq 0$.)

Tod. Kirjoitetaan $q_1 = \alpha + u$, $q_2 = \beta + v$, $\alpha, \beta \in \mathbb{R}$, $\operatorname{Re} u, \operatorname{Re} v = 0$

$$q_1 q_2 = q_2 q_1 \iff u \cdot v = v \cdot u$$

$$\underset{T.1.2}{\iff} u \times v = \underbrace{v \times u}_{= -u \times v} \quad \text{kun } u, v \text{ tulkitaan } \mathbb{R}^3\text{:n vektoreiksi}$$

$$\iff u \times v = 0$$

$$\iff u = \lambda v \quad \text{jollakin } \lambda \in \mathbb{R}. \qquad \square$$

(1.4) Olkoon $\begin{cases} q_1 = \alpha + u \\ q_2 = \beta + v \end{cases}$, $\alpha, \beta \in \mathbb{R}$, $u, v \in \mathbb{H}$, $\operatorname{Re} u, \operatorname{Re} v = 0$

Tällöin $q_1 q_2 = -q_2 q_1 \iff$ (a) $q_1 = 0$ tai/ja $q_2 = 0$

(b) $\alpha = \beta = 0$, $\langle u, v \rangle_{\mathbb{R}^3} = 0$

Tod. $q_1 q_2 = \alpha\beta + \alpha v + \beta u + uv$

$$\Rightarrow q_1 q_2 + q_2 q_1 = 0 \iff 2\alpha\beta + 2\alpha v + 2\beta u + uv + vu = 0$$

$$\iff \left[\alpha\beta = \langle u, v \rangle_{\mathbb{R}^3},\ \alpha v + \beta u = 0\right] = (*)$$

$$\left[\alpha = \beta = 0,\ (*)\right] = E_1$$

$$\iff \text{tai } \left[\text{Joko } \alpha = 0, \text{ tai } \beta = 0,\ (*)\right] = E_2$$

$$\text{tai } \left[\alpha, \beta \neq 0,\ (*)\right] = E_3$$

$E_1 \iff [\alpha = \beta = 0,\ \langle u, v \rangle]$;

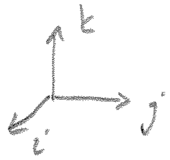
$E_2 \iff [\alpha = 0, \beta \neq 0, \langle u, v \rangle = 0, \beta u = 0]$ tai $[\beta = 0, \alpha \neq 0, \langle u, v \rangle = 0, \alpha v = 0]$

$\iff [q_1 = 0]$ tai $[q_2 = 0]$

$$E_3 \iff \Big[\alpha, \beta \neq 0,\ \underbrace{\alpha\beta = \langle u, v \rangle,\ v = -\frac{\beta}{\alpha} u}_{\Rightarrow \alpha^2 = -|u|^2}\Big] \quad \lightning \qquad \square$$

1.5 a) Jos $qi = iq$, $q \in \mathbb{H}$, niin $\underbrace{q \in \mathbb{C}}$.

Jos $q = a + bi + cj + dk$ $\left(\begin{array}{l} = (a,b,0,0) \\ \text{joillakin} \\ a,b \in \mathbb{R} \end{array} \right)$



$$\Rightarrow \quad iq = ai - b + ck - dj$$
$$qi = ai - b - ck + dj$$
$$\Rightarrow \quad ck - dj = 0 \qquad |\cdot|$$
$$\Rightarrow \quad c^2 + d^2 = 0 \qquad \Rightarrow \quad q = a + bi \in \mathbb{C} \subset \mathbb{H}.$$

b) Jos $\lambda \in \mathbb{H}$ ja $\lambda q = q\lambda \quad \forall q \in \mathbb{H}$ niin $\lambda \in \mathbb{R}$.

$(q=i)$ $\quad i\lambda = \lambda i \quad \Rightarrow \quad \lambda = a + bi, \quad a,b \in \mathbb{R}$

$(q=j)$ $\quad j(a+bi) = aj - bk = (a+bi)j = aj + bk$

$$\Rightarrow \quad bk = 0 \quad \Rightarrow \quad b = 0$$
$$\Rightarrow \quad \lambda \in \mathbb{R}. \qquad \square$$

1.6

Tarkastellaan kuvausta $\quad L: \mathbb{H}^2 \longrightarrow \mathbb{H}^2$

$$(u,v) \longmapsto (u,v)\begin{pmatrix} i & j \\ i & j \end{pmatrix}$$

a) $L((0,0)) = (0,0)\begin{pmatrix} i & j \\ i & j \end{pmatrix} = (0,0)$

$L((1,-1)) = (1,-1)\begin{pmatrix} i & j \\ i & j \end{pmatrix} = (0,0) \rightsquigarrow L$ ei injektio

b) Laajennetaan $\Psi: \mathbb{H} \to M_{2\times 2}(\mathbb{C})$ kuvaukseksi

$$\Psi: M_{n\times m}(\mathbb{H}) \longrightarrow M_{2n \times 2m}(\mathbb{C}) \qquad 2\times 2\ \mathbb{C}\text{-lohko}$$

$$a_{ij} \longmapsto \left[\begin{array}{c|c|c} \Psi(a_{11}) & \Psi(a_{12}) & \cdots \\ \hline \Psi(a_{21}) & \ddots & \\ \hline \vdots & \vdots & \end{array}\right]$$

Pätee $\quad \Psi(ax + by) = \Psi(a)\Psi(x) + \Psi(b)\Psi(y) \quad a,b,x,y \in \mathbb{H}$

joten $\quad \Psi(v \cdot M) = \Psi(v) \cdot \Psi(M) \qquad v \in M_{1\times n}(\mathbb{H}), M \in M_{n\times m}(\mathbb{H})$

Nyt on luonnollista määritellä

$$\det M = \det \Psi(M), \qquad M \in M_{n\times n}(\mathbb{H})$$

$$\det \underbrace{(a,b,c,d)}_{\in \mathbb{H}} = \det \begin{pmatrix} a+ib & c+id \\ -\overline{(c+id)} & \overline{a+ib} \end{pmatrix}$$

$$= a^2+b^2+c^2+d^2$$

eli $\det u = |u|^2$

$$\det \begin{pmatrix} i & j \\ i & j \end{pmatrix} = \det \left(\begin{array}{c|c} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \hline \underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}_{\Psi(i)} & \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\Psi(j)} \end{array} \right) = 0$$

Cramer'n säännöllä saadaan:

$$\det \begin{pmatrix} i & j \\ i & j \end{pmatrix} = ij - ji = 2k$$ (kehitetty 1:n rivin suhteen)

tai $$\det \begin{pmatrix} i & j \\ i & j \end{pmatrix} = ij - ij = 0.$$ (kehitetty 1:n sarakkeen suhteen)