TKK Matematiikan laitos Arponen/Pursiainen

Mat-1.3651 Numerical Linear Algebra, spring 2008
(Numeerinen matriisilaskenta, kevät 2008)

Exercise 2 (31.1.2008)

Please hand in the exercises marked with an asterisk (*) either to the assistant's folder in front of U313 or latest at the beginning of the exercise.

* 1. Let W be an invertible matrix. Show that the function $\|\cdot\|_W$ defined by

$$\|x\|_W := \|Wx\|$$

is a vector norm.

2. Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show that $\rho(A) \leq \|A\|$, where $\rho(A) = \max|\lambda|$ is the *spectral radius* of A, i.e. the largest absolute value of an eigenvalue λ of A.

3. Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$. The corresponding *dual norm* $\|\cdot\|'$ is defined by the formula $\|x\|' = \sup_{\|y\|=1} |y^*x|$.

   (a) Prove that $\|\cdot\|'$ is a norm.

   (b) Let $x, y \in \mathbb{C}^m$ with $\|x\| = \|y\| = 1$ be given. Show that there exists a rank-one matrix $B = yz^*$ such that $Bx = y$ and $\|B\| = 1$, where $\|B\|$ is the matrix norm of B induced by the vector norm $\|\cdot\|$. You may assume the following lemma known: given $x \in \mathbb{C}^m$, there exists a nonzero $z \in \mathbb{C}^m$ s.t. $|z^*x| = \|z\|'\|x\|$.

* 4. Determine SVDs of the following matrices, by hand calculation:

$$\text{(a)} \quad \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}, \text{(b)} \quad \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{(c)} \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

* 5. In an example on the lecture we claimed that for the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

the 2-norm is $\|A\|_2 \approx 2.9208$. Using the SVD calculate $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ and deduce $\|A\|_2$.