Helsinki University of Technology Institute of Mathematics Research Reports
Espoo 2010 A586

MINIMAL RESIDUAL METHODS FOR SOLVING A CLASS OF $\mathbb{R}$-LINEAR SYSTEMS OF EQUATIONS

Kui Du Olavi Nevanlinna

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

Helsinki University of Technology Institute of Mathematics Research Reports
Espoo 2010 A586

MINIMAL RESIDUAL METHODS FOR SOLVING A CLASS OF $\mathbb{R}$-LINEAR SYSTEMS OF EQUATIONS

Kui Du **Olavi Nevanlinna**

Aalto University
School of Science and Technology
Department of Mathematics and Systems Analysis

Kui Du, Olavi Nevanlinna: *Minimal residual methods for solving a class of $\mathbb{R}$-linear systems of equations*; Helsinki University of Technology Institute of Mathematics Research Reports A586 (2010).

Abstract: *Recently, $\mathbb{R}$-linear GMRES was proposed by Eirola, Huhtanen and von Pfaler [SIAM J. Matrix Anal. Appl. 25 (2004), pp. 804–828] for solving a class of $\mathbb{R}$-linear systems of equations. In this work we investigate $\mathbb{R}$-linear GMRES through the equivalent real formulations of the $\mathbb{R}$-linear system. We show that $\mathbb{R}$-linear GMRES requires fewer matrix-vector products than GMRES applied to the related $\mathbb{C}$-linear system. Numerical results for an artificial example and the inverse problem of reconstructing an unknown electric conductivity are reported to confirm our theoretical results.*

AMS subject classifications: 65F10, 15A18

Keywords: $\mathbb{R}$-linear GMRES, equivalent real formulation, spectral analysis

Correspondence

Kui Du
Aalto University
Department of Mathematics and Systems Analysis
P.O. Box 11100
FI-00076 Aalto
Finland
kuidumath@yahoo.com

Olavi Nevanlinna
Aalto University
Department of Mathematics and Systems Analysis
P.O. Box 11100
FI-00076 Aalto
Finland
olavi.nevanlinna@tkk.fi

Received 2010-05-06

ISBN 978-952-60-3181-1 (print) ISSN 0784-3143 (print)
ISBN 978-952-60-3182-8 (PDF) ISSN 1797-5867 (PDF)

Aalto University
School of Science and Technology
Department of Mathematics and Systems Analysis
P.O. Box 11100, FI-00076 Aalto, Finland
email: math@tkk.fi `http://math.tkk.fi/`

MINIMAL RESIDUAL METHODS FOR SOLVING A CLASS OF $\mathbb{R}$-LINEAR SYSTEMS OF EQUATIONS

KUI DU† AND OLAVI NEVANLINNA†

Abstract. Recently, $\mathbb{R}$-linear GMRES was proposed by Eirola, Huhtanen and von Pfaler [*SIAM J. Matrix Anal. Appl.* 25 (2004), pp. 804-828] for solving a class of $\mathbb{R}$-linear systems of equations. In this work we investigate $\mathbb{R}$-linear GMRES through the equivalent real formulations of the $\mathbb{R}$-linear system. We show that $\mathbb{R}$-linear GMRES requires fewer matrix-vector products than GMRES applied to the related $\mathbb{C}$-linear system. Numerical results for an artificial example and the inverse problem of reconstructing an unknown electric conductivity are reported to confirm our theoretical results.

Key words. $\mathbb{R}$-linear GMRES, equivalent real formulation, spectral analysis

AMS subject classifications. 65F10, 15A18

1. Introduction. Consider the $\mathbb{R}$-linear system

$$\kappa z + M\bar{z} = b, \tag{1.1}$$

where $\kappa \in \mathbb{C}, z, b \in \mathbb{C}^n, M \in \mathbb{C}^{n\times n}$ and $\bar{z}$ denotes the conjugate of z. We use $\mathcal{M}_\kappa z = (\kappa I + M\tau)z$ to denote the left-hand side, where I denotes the identity matrix whose dimension is clear from the context and $\tau z = \bar{z}$ is the conjugation operator on $\mathbb{C}^n$. Such $\mathbb{R}$-linear systems arise in the inverse problem of reconstructing an unknown electric conductivity in the unit disc from boundary measurements [18, 14, 1], especially in the numerical discretization of the $\mathbb{R}$-linear Beltrami equation [11] and the $\bar{\partial}$-equation [13]. For theoretical analysis of general $\mathbb{R}$-linear operators $\mathcal{A} = M + N\tau$ where $M, N \in \mathbb{C}^{n\times n}$, we refer to [6, 9, 10, 12].

By rewriting (1.1) as an equivalent real formulation of doubled size for its real and imaginary parts, any standard Krylov subspace method [16, 21] can de used to solve the problem. Equivalent real formulations of a $\mathbb{C}$-linear (complex linear) system have been considered by several researchers, see, for example, [7, 2, 4, 3]. Recently, the $\mathbb{R}$-linear GMRES (RL-GMRES) method [6] was proposed for solving (1.1), which avoids using an equivalent real formulation. In this paper, we investigate RL-GMRES through the equivalent real formulation of (1.1).

Huhtanen and Perämäki [11] considered preconditioning techniques for the $\mathbb{R}$-linear system (1.1). By the right preconditioner $\bar{\kappa}I - M\tau$, they obtained the $\mathbb{C}$-linear system

$$|\kappa|^2 w - M\overline{M}w = b. \tag{1.2}$$

If (1.2) is solved, then z can be readily obtained by $z = \bar{\kappa}w - M\bar{w}$. In this paper, we show that RL-GMRES applied to (1.1) is faster than GMRES [17] applied to (1.2) in terms of matrix-vector products, namely, RL-GMRES requires fewer matrix-vector products; see Remark 3.10.

The paper is organized as follows. In section 2 we discuss equivalent real formulations of (1.1). Spectral properties of the equivalent real formulations of the $\mathbb{R}$-linear operator $\mathcal{M}_\kappa$ are studied. In section 3 we review RL-GMRES and investigate it through the equivalent real formulation. In section 4 we report numerical experiments confirming our theoretical results. In the last section we present our conclusions.

†Institute of Mathematics, Aalto University, P.O.Box 11100, FI-00076 Aalto, Finland (kuidumath@yahoo.com, Olavi.Nevanlinna@tkk.fi).

2. Equivalent real formulations. Write $\kappa = \alpha + \mathrm{i}\beta$, $z = x + \mathrm{i}y$, $b = c + \mathrm{i}d$ and $M = A + \mathrm{i}B$, where $\mathrm{i} = \sqrt{-1}$, $\alpha, \beta \in \mathbb{R}$, $x, y, c, d \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n\times n}$. The $\mathbb{R}$-linear system (1.1) admits several $2n \times 2n$ equivalent real formulations. We list four representative ones in (2.1)-(2.4), and call them R1 to R4, respectively.

R1 formulation.

$$\left(\begin{bmatrix} A & -B \\ B & A \end{bmatrix} + \alpha \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \beta \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\right) \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}. \tag{2.1}$$

R2 formulation.

$$\left(\begin{bmatrix} A & B \\ B & -A \end{bmatrix} + \alpha \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \beta \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}\right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}. \tag{2.2}$$

R3 formulation.

$$\left(\begin{bmatrix} B & A \\ A & -B \end{bmatrix} + \alpha \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} + \beta \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\right) \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} d \\ c \end{bmatrix}. \tag{2.3}$$

R4 formulation.

$$\left(\begin{bmatrix} B & -A \\ A & B \end{bmatrix} + \alpha \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} + \beta \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d \\ c \end{bmatrix}. \tag{2.4}$$

For notational simplicity, let

$$\mathbf{M}_1 = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}, \mathbf{M}_2 = \begin{bmatrix} A & B \\ B & -A \end{bmatrix}, \mathbf{M}_3 = \begin{bmatrix} B & A \\ A & -B \end{bmatrix}, \mathbf{M}_4 = \begin{bmatrix} B & -A \\ A & B \end{bmatrix}, \tag{2.5}$$

and

$$\mathbf{I} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \tag{2.6}$$

Let $\|\cdot\|$ denote the Euclidean vector norm or the associated induced matrix norm. We have

$$\|\mathbf{M}_1\| = \|\mathbf{M}_2\| = \|\mathbf{M}_3\| = \|\mathbf{M}_4\| = \|M\|. \tag{2.7}$$

Let $\mathbf{R}_1$ to $\mathbf{R}_4$ denote the matrices associated with the four formulations. We have

$$\mathbf{R}_1 = \mathbf{M}_1 + \alpha\mathbf{F} + \beta\mathbf{E}, \qquad \mathbf{R}_2 = \mathbf{M}_2 + \alpha\mathbf{I} + \beta\mathbf{J}, \tag{2.8}$$

$$\mathbf{R}_3 = \mathbf{M}_3 + \alpha\mathbf{J} + \beta\mathbf{I}, \qquad \mathbf{R}_4 = \mathbf{M}_4 + \alpha\mathbf{E} + \beta\mathbf{F}. \tag{2.9}$$

2.1. Spectral properties. Let $\sigma(M)$ denote the spectrum of M, $\overline{\sigma(M)}$ the set of complex conjugates of the elements of $\sigma(M)$ and $\overline{M}$ the componentwise complex conjugate of M. The matrix $M\overline{M}$ is important in our analysis. Its spectrum, $\sigma(\overline{M}M)$, is symmetric with respect to the real axis, i.e., if $\mu \in \sigma(\overline{M}M)$, then $\bar{\mu} \in \sigma(\overline{M}M)$. Moreover, the eigenvalues μ and $\bar{\mu}$ are of the same multiplicity. The matrix $\overline{M}M$ is similar to $M\overline{M}$. We refer the reader to [8, Section 4.6] for the proofs of these properties.

Lemma 2.1. *Let* $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$ *and* $\mathbf{J}$ *be the matrices in* (2.5) *and* (2.6). *Assume* $\sigma(M\overline{M}) = \{\mu_i, i = 1, \ldots, m\}$. *Then we have the following facts:*

(1) $\sigma(\mathbf{M}_1) = \sigma(M) \cup \overline{\sigma(M)}$, $\sigma(\mathbf{M}_4) = -\mathrm{i}\sigma(M) \cup \mathrm{i}\overline{\sigma(M)}$.

(2) *If* $\lambda \in \sigma(\mathbf{M}_2)$*, then* $-\lambda, \bar{\lambda}, -\bar{\lambda} \in \sigma(\mathbf{M}_2)$.
(3) *If* $\lambda \in \sigma(\mathbf{M}_3)$*, then* $-\lambda, \bar{\lambda}, -\bar{\lambda} \in \sigma(\mathbf{M}_3)$.
(4) $\sigma(\mathbf{M}_2) = \sigma(\mathbf{J}\mathbf{M}_2) = \sigma(\mathbf{M}_3) = \sigma(\mathbf{J}\mathbf{M}_3) = \{\pm\sqrt{\mu_i}, i = 1, \dots, m\}$.

Proof. See Proposition 5.1(a)(b) of [7] and Section 1.1 of [4]. □

LEMMA 2.2. *Let* $\omega \in \mathbb{C}$*,* $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4, \mathbf{E}, \mathbf{J}$ *be the matrices in* (2.5) *and* (2.6). *Assume* $\sigma(M\overline{M}) = \{\mu_i, i = 1, \dots, m\}$. *Then we have the following facts:*
(1) $\sigma(\mathbf{M}_1 + \omega\mathbf{E}) = \sigma(\mathbf{M}_1 - \omega\mathbf{E})$, $\sigma(\mathbf{M}_4 + \omega\mathbf{E}) = \sigma(\mathbf{M}_4 - \omega\mathbf{E})$.
(2) $\sigma(\mathbf{M}_2 + \omega\mathbf{J}) = \sigma(\mathbf{M}_2 - \omega\mathbf{J}) = \sigma(\mathbf{M}_3 + \omega\mathbf{J}) = \sigma(\mathbf{M}_3 - \omega\mathbf{J}) = \{\pm\sqrt{\mu_i - \omega^2}, i = 1, \dots, m\}$.

Proof. By $-\mathbf{J}\mathbf{J} = \mathbf{I}$, $-\mathbf{J}(\mathbf{M}_1 + \omega\mathbf{E})\mathbf{J} = \mathbf{M}_1 - \omega\mathbf{E}$, and $-\mathbf{J}(\mathbf{M}_4 + \omega\mathbf{E})\mathbf{J} = \mathbf{M}_4 - \omega\mathbf{E}$, (1) is obvious. Let

$$\mathbf{S} = \frac{\sqrt{2}}{2}\begin{bmatrix} I & -\mathrm{i}I \\ -\mathrm{i}I & I \end{bmatrix}. \tag{2.10}$$

By a straightforward calculation, we have

$$\mathbf{S}(\mathbf{M}_2 + \omega\mathbf{J})^2\mathbf{S}^* = \begin{bmatrix} \overline{M}M - \omega^2 I & 0 \\ 0 & M\overline{M} - \omega^2 I \end{bmatrix}, \tag{2.11}$$

and

$$(\mathbf{M}_2 + \omega\mathbf{J})^2 = (\mathbf{M}_2 - \omega\mathbf{J})^2 = \mathbf{M}_2^2 - \omega^2\mathbf{I}. \tag{2.12}$$

By (2.11), (2.12), Lemma 2.1 (4) and $\sigma(M\overline{M}) = \sigma(\overline{M}M)$, we have

$$\sigma(\mathbf{M}_2 + \omega\mathbf{J}) = \sigma(\mathbf{M}_2 - \omega\mathbf{J}) = \{\pm\sqrt{\mu_i - \omega^2}, i = 1, \dots, m\}.$$

Similarly,

$$\sigma(\mathbf{M}_3 + \omega\mathbf{J}) = \sigma(\mathbf{M}_3 - \omega\mathbf{J}) = \{\pm\sqrt{\mu_i - \omega^2}, i = 1, \dots, m\}.$$

We obtain (2). □

THEOREM 2.3. *Let* $\alpha, \beta \in \mathbb{R}$*,* $\mathbf{R}_2, \mathbf{R}_3$ *be the matrices in* (2.8) *and* (2.9). *Assume* $\sigma(M\overline{M}) = \{\mu_i, i = 1, \dots, m\}$. *Then we have the following facts:*
(1) $\sigma(\mathbf{R}_2) = \{\alpha \pm \sqrt{\mu_i - \beta^2}, i = 1, \dots, m\}$ *is symmetric with respect to the point* $(\alpha, 0)$ *and the real axis.*
(2) $\sigma(\mathbf{R}_3) = \{\beta \pm \sqrt{\mu_i - \alpha^2}, i = 1, \dots, m\}$ *is symmetric with respect to the point* $(\beta, 0)$ *and the real axis.*

Proof. (1) and (2) are direct results of Lemma 2.2 (2) and the eigenvalues come in conjugate pairs for real matrices. □

3. $\mathbb{R}$-linear GMRES. In this section, we investigate the $\mathbb{R}$-linear GMRES for solving (1.1) through the R2 formulation (2.2). Let

$$\mathcal{K}_i(\mathcal{M}_\kappa, b) := \mathrm{span}\{b, \mathcal{M}_\kappa b, \dots, \mathcal{M}_\kappa^{i-1}b\} \subset \mathbb{C}^n,$$

denote the ith Krylov subspace generated by $\mathcal{M}_\kappa$ and $b \in \mathbb{C}^n$. Define $\phi : \mathbb{C}^n \to \mathbb{R}^{2n}$ by

$$\phi(z) = \begin{bmatrix} \mathrm{Re}(z) \\ \mathrm{Im}(z) \end{bmatrix}.$$

Let

$$\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b)) := \operatorname{span}^{\mathbb{R}}\{\phi(b), \mathbf{R}_2\phi(b), \dots, \mathbf{R}_2^{i-1}\phi(b)\} \subset \mathbb{R}^{2n},$$

denote the ith Krylov subspace generated by $\mathbf{R}_2$ and $\phi(b) \in \mathbb{R}^{2n}$. Here and in the sequel, the notation "$\operatorname{span}^{\mathbb{R}}\{\cdots\}$" denotes the space of all real linear combinations of the vectors in braces. We list some important properties of the Krylov subspaces $\mathcal{K}_i(\mathcal{M}_\kappa, b)$ and $\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b))$ in the following lemma. The proof is easy, and is therefore omitted here.

Lemma 3.1. *Let $\mathcal{M}_\kappa = \kappa I + M\tau$, $b \in \mathbb{C}^n$, $\mathbf{R}_2$ and $\mathbf{J}$ be the matrices in* (2.8) *and* (2.6)*. Then we have the following facts:*

(1) $\mathcal{K}_i(\mathcal{M}_\kappa, b) = \operatorname{span}\{b, M\tau b, \dots, (M\tau)^{i-1}b\}$;
(2) $\mathcal{K}_i(\mathcal{M}_\kappa, b) \subseteq \mathcal{K}_{i+1}(\mathcal{M}_\kappa, b)$;
(3) $\mathcal{M}_\kappa(\mathcal{K}_i(\mathcal{M}_\kappa, b)) \subseteq \mathcal{K}_{i+1}(\mathcal{M}_\kappa, b)$;
(4) $\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b)) \subset \phi(\mathcal{K}_i(\mathcal{M}_\kappa, b)), \mathbf{J}\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b)) \subset \phi(\mathcal{K}_i(\mathcal{M}_\kappa, b))$;
(5) $\phi(\mathcal{K}_i(\mathcal{M}_\kappa, b)) = \mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b)) + \mathbf{J}\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b))$;
(6) $\operatorname{span}\{b\} + \mathcal{M}_\kappa(\mathcal{K}_i(\mathcal{M}_\kappa, b)) = \mathcal{K}_{i+1}(\mathcal{M}_\kappa, b)$;
(7) $\operatorname{span}^{\mathbb{R}}\{\phi(b), \mathbf{J}\phi(b)\} + \mathbf{R}_2\phi(\mathcal{K}_i(\mathcal{M}_\kappa, b)) = \phi(\mathcal{K}_{i+1}(\mathcal{M}_\kappa, b))$;
(8) *There is a positive integer m such that*

$$\dim(\mathcal{K}_i(\mathcal{M}_\kappa, b)) = \begin{cases} i, & 1 \le i \le m, \\ m, & i \ge m+1, \end{cases}$$

$$\dim(\phi(\mathcal{K}_i(\mathcal{M}_\kappa, b))) = \begin{cases} 2i, & 1 \le i \le m, \\ 2m, & i \ge m+1, \end{cases}$$

*where "*dim*" denotes dimension.*

We remark that $\mathbf{R}_2\phi(\mathcal{K}_i(\mathcal{M}_\kappa, b))$ is a linear subspace of $\mathbb{R}^{2n}$ and $\mathcal{M}_\kappa(\mathcal{K}_i(\mathcal{M}_\kappa, b))$ is an $\mathbb{R}$-linear subspace of $\mathbb{C}^n$. We can generate the Arnoldi basis of $\mathcal{K}_i(\mathcal{M}_\kappa, b)$ by the Arnoldi process [16, 21].

Let z_0 be the initial guess and $r_0 = b - \mathcal{M}_\kappa z_0$ the corresponding residual. The ith iterate, z_i, determined by RL-GMRES satisfies

$$\|b - \mathcal{M}_\kappa z_i\| = \min_{w \in z_0 + \mathcal{K}_i(\mathcal{M}_\kappa, r_0)} \|b - \mathcal{M}_\kappa w\|, \qquad z_i \in z_0 + \mathcal{K}_i(\mathcal{M}_\kappa, r_0).$$

Let $\widetilde{I}_i$ denote the $i \times i$ identity matrix augmented with the row of zeros as the last row and e_1 the first column of the identity matrix with appropriate dimension. Let $H_{i+1,i}$ be the upper Hessenberg matrix generated in the Arnoldi process (see step 2 of Algorithm 1 below). The ith iterate z_i satisfies

$$\|b - \mathcal{M}_\kappa z_i\| = \min_{s \in \mathbb{C}^i} \left\| \|r_0\| e_1 - \kappa \widetilde{I}_i s - H_{i+1,i}\bar{s} \right\|.$$

The minimal problem

$$\min_{s \in \mathbb{C}^i} \left\| \|r_0\| e_1 - \kappa \widetilde{I}_i s - \widetilde{H}_{i+1,i}\bar{s} \right\|$$

can be solved by employing the $\mathbb{R}$-linear QR decomposition [6]. The work and storage of RL-GMRES (as a function of the number of iterations) are comparable to those of GMRES. We give the details of RL-GMRES in Algorithm 1.

Algorithm 1: **$\mathbb{R}$-linear GMRES**

1. Compute $r_0 = b - \mathcal{M}_\kappa z_0$, z_0 is the initial guess
2. Generate the Arnoldi basis and the matrix $H_{i+1,i}$:
   - $v_1 = r_0/\|r_0\|$;
   - **for** $j = 1, 2, \ldots,$ **do**
     - $w = M\bar{v}_j$
     - **for** $i = 1$ to j **do**
       - $h_{ij} = v_i^* w$
       - $w = w - h_{ij} v_i$
     - **end for**
     - $h_{j+1,j} = \|w\|$
     - $v_{j+1} = w/h_{j+1,j}$
     - Solve the minimal problem $\min_{s\in\mathbb{C}^i} \left\| \|r_0\| e_1 - \kappa \widetilde{I}_i s - H_{i+1,i}\bar{s} \right\|$ for s
     - Set $z_i = z_0 + V_i s$ and $r_i = b - \mathcal{M}_\kappa z_i$
     - Exit if satisfied
   - **end for**

For notational simplicity, in the sequel, we choose the initial guess to be $z_0 = 0$. In step 2 of Algorithm 1, application of i steps of the Arnoldi process with starting vector b yields the Arnoldi decomposition

$$\mathcal{M}_\kappa V_i = V_{i+1}(\kappa \widetilde{I}_i + H_{i+1,i}), \tag{3.1}$$

where $V_i = [v_1, v_2, \ldots, v_i]$ and $V_i^* V_i = I$. The columns of V_i form an orthonormal basis of the Krylov subspace $\mathcal{K}_i(\mathcal{M}_\kappa, b)$; see [6, Theorem 3.1].

We say that the Arnoldi process (3.1) breaks down at step m if $h_{m+1,m} = 0$. Similar to that of GMRES, the exact solution is determined by RL-GMRES when breakdown occurs. However, the proof is not trivial because in general $\mathcal{M}_\kappa(\mathcal{K}_i(\mathcal{M}_\kappa, b))$ is not a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$. When $h_{m+1,m} = 0$,

$$\mathcal{M}_\kappa V_m = V_m(\kappa I + H_m),$$

where H_m is the leading $m \times m$ submatrix of $H_{m+1,m}$. We have

$$\dim(\mathcal{K}_{m+1}(\mathcal{M}_\kappa, b)) = \dim(\mathcal{K}_m(\mathcal{M}_\kappa, b)) = m,$$

and

$$\dim(\phi(\mathcal{K}_{m+1}(\mathcal{M}_\kappa, b))) = \dim(\phi(\mathcal{K}_m(\mathcal{M}_\kappa, b))) = 2m. \tag{3.2}$$

If $\mathbf{R}_2$ is nonsingular, then it follows from $\dim(\mathbf{R}_2\phi(\mathcal{K}_m(\mathcal{M}_\kappa, b))) = 2m$, (3.2) and (7) of Lemma 3.1 that

$$\operatorname{span}^{\mathbb{R}}\{\phi(b), \mathbf{J}\phi(b)\} \subseteq \mathbf{R}_2\phi(\mathcal{K}_m(\mathcal{M}_\kappa, b)).$$

Therefore,

$$\operatorname{span}\{b\} \subseteq \mathcal{M}_\kappa(\mathcal{K}_m(\mathcal{M}_\kappa, b)). \tag{3.3}$$

It follows from (3.3) that

$$\|r_m\| = \|b - \mathcal{M}_\kappa z_m\| = 0.$$

This is referred to as a *benign* breakdown.

In the sequel, $\mathbb{P}_i$ denotes the set of all complex polynomials of degree at most i, and $\mathbb{P}_i^{\mathbb{R}}$ denotes the subset of $\mathbb{P}_i$ of polynomials with real coefficients. The RL-GMRES process chooses the coefficients of $p_{i-1} \in \mathbb{P}_{i-1}$ to minimize the norm of the residual $r_i = b - \mathcal{M}_\kappa(p_{i-1}(\mathcal{M}_\kappa)b)$. The proof of the following proposition is obvious.

PROPOSITION 3.2. *Let* $\mathbf{J}$ *and* $\mathbf{R}_2$ *be the matrices in* (2.6) *and* (2.8)*, and* r_i *be the* ith *residual of* RL-GMRES *applied to* (1.1)*. Then*

$$\|r_i\| = \min_{w \in \mathcal{K}_i(\mathcal{M}_\kappa, b)} \|b - \mathcal{M}_\kappa w\| = \min_{\phi(w) \in \phi(\mathcal{K}_i(\mathcal{M}_\kappa, b))} \|\phi(b) - \mathbf{R}_2 \phi(w)\| \tag{3.4}$$

$$= \min_{p_1, p_2 \in \mathbb{P}_{i-1}^{\mathbb{R}}} \|\phi(b) - \mathbf{R}_2 p_1(\mathbf{R}_2)\phi(b) - \mathbf{R}_2 \mathbf{J} p_2(\mathbf{R}_2)\phi(b)\|. \tag{3.5}$$

Let $r_i^{\mathbb{R}}$ be the ith residual of GMRES applied to (2.2), we have

$$\|r_i^{\mathbb{R}}\| = \min_{u \in \mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b))} \|\phi(b) - \mathbf{R}_2 u\|. \tag{3.6}$$

By (3.4) and (3.6), RL-GMRES can be viewed as an augmented GMRES method. At step i, the subspace $\mathbf{J}\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b))$ is added. It follows from $\mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b)) \subseteq \phi(\mathcal{K}_i(\mathcal{M}_\kappa, b))$ that

$$\|r_i\| \le \|r_i^{\mathbb{R}}\|. \tag{3.7}$$

This inequality implies that RL-GMRES applied to (1.1) is better than GMRES applied to (2.2). See also Proposition 3.6 of [6]. A sufficient condition for GMRES applied to (2.2) having the same performance as RL-GMRES applied to (1.1) is given in Proposition 3.3. The proof of this proposition is obvious.

PROPOSITION 3.3. *If* $\mathcal{K}_{i+1}^{\mathbb{R}}(\mathbf{R}_2, \phi(b)) \perp \mathbf{R}_2 \mathbf{J} \mathcal{K}_i^{\mathbb{R}}(\mathbf{R}_2, \phi(b))$ *then* $\|r_i\| = \|r_i^{\mathbb{R}}\|$.

Similar to the GMRES approximation problem [20, p.269], we can define the RL-GMRES approximation problem

$$\text{Find } p_{i-1} \in \mathbb{P}_{i-1} \text{ such that } \|b - \mathcal{M}_\kappa(p_{i-1}(\mathcal{M}_\kappa)b)\| = \text{minimum}. \tag{3.8}$$

THEOREM 3.4.

(1) RL-GMRES *is scale-invariant, i.e., if* $\mathcal{M}_\kappa$ *is changed to* $\omega\mathcal{M}_\kappa$ *for* $\omega \neq 0 \in \mathbb{C}$*, and* b *is changed to* ωb*, the residuals* $\{r_i\}$ *change to* $\{\omega r_i\}$.

(2) RL-GMRES *is invariant under unitary similarity transformations, i.e., if* $\mathcal{M}_\kappa$ *is changed to* $U\mathcal{M}_\kappa U^*$ *for some unitary matrix* U*, and* b *is changed to* Ub*, the residuals* $\{r_i\}$ *change to* $\{Ur_i\}$.

Proof. By (1) of Lemma 3.1, for any $p_{i-1} \in \mathbb{P}_{i-1}$, there exist $\tilde{p}_{i-1}, \hat{p}_{i-1}, q_{i-1} \in \mathbb{P}_{i-1}$ such that $p_{i-1}(\omega\mathcal{M}_\kappa)(\omega b) = \tilde{p}_{i-1}(\omega M\tau)(\omega b) = \hat{p}_{i-1}(M\tau)b = q_{i-1}(\mathcal{M}_\kappa)b$. Then (1) is readily proved from (3.8). The proof of (2) is easy. □

In view of Theorem 3.4 (1) and Proposition 3.2, if one matrix of the set

$$\{\widetilde{\mathbf{M}}_2 + \tilde{\alpha}\mathbf{I} + \tilde{\beta}\mathbf{J} : \tilde{\alpha}, \tilde{\beta} \in \mathbb{R}, \tilde{\alpha}^2 + \tilde{\beta}^2 = |\kappa|^2\}$$

has desirable properties for GMRES convergence rate, then RL-GMRES applied to (1.1) has speedy convergence rate (see Example 2 of section 4). Here $\tilde{\alpha} + \mathrm{i}\tilde{\beta} = e^{\mathrm{i}\vartheta}(\alpha + \mathrm{i}\beta)$, $\vartheta \in [0, 2\pi)$, $\widetilde{\mathbf{M}}_2 = \Theta\mathbf{M}_2$, and $\Theta = \cos\vartheta\mathbf{I} + \sin\vartheta\mathbf{J}$.

Let $\mathcal{N} = P + Q\tau$, where $P, Q \in \mathbb{C}^{n\times n}$. We have

$$\mathcal{M}_\kappa\mathcal{N} = (\kappa I + M\tau)(P + Q\tau) = (\kappa P + M\overline{Q}) + (\kappa Q + M\overline{P})\tau.$$

To use RL-GMRES, the preconditioned $\mathbb{R}$-linear operator $\mathcal{M}_\kappa\mathcal{N}$ should preserve the form $\omega I + N\tau$, i.e., $\kappa P + M\overline{Q} = \omega I$. On the other hand, if $\kappa Q + M\overline{P} = 0$ then the preconditioned $\mathbb{R}$-linear operator $\mathcal{M}_\kappa\mathcal{N} = \kappa P + M\overline{Q}$. Simplest options for choosing P are diagonal matrices. Choosing $P = \bar{\kappa}I$ and $Q = -M$, we have the linear system (1.2) with $z = (\bar{\kappa}I - M\tau)w$.

To compare RL-GMRES applied to (1.1) with GMRES applied to (1.2), we need the following lemma which describes the shift-invariance property [19] of Krylov subspaces.

Lemma 3.5. *For any $A \in \mathbb{R}^{n\times n}$, $\rho \in \mathbb{R}$, $x \in \mathbb{R}^n$,*

$$\mathcal{K}_i^{\mathbb{R}}(A + \rho I, x) = \mathcal{K}_i^{\mathbb{R}}(A, x),$$

i.e., any $u = p_{i-1}(A+\rho I)x$ with $p_{i-1} \in \mathbb{P}_{i-1}^{\mathbb{R}}$ can be expressed as $u = \tilde{p}_{i-1}(A)x$, where $\tilde{p}_{i-1}(\zeta) = p_{i-1}(\zeta + \rho)$.

In the following proposition, the residual norms of RL-GMRES applied to (1.1) and GMRES applied to (1.2) are expressed by the equivalent real formulations.

Proposition 3.6. *Let $\mathbf{M}_2$, $\mathbf{I}$ be the matrices in* (2.5)-(2.6), $\kappa = |\kappa|e^{\mathrm{i}\theta}$, $\Theta = \cos\theta\mathbf{I} - \sin\theta\mathbf{J}$, $\mathbf{M} = \Theta\mathbf{M}_2$ *and* $b_\theta = e^{-\mathrm{i}\theta}b$.

(1) *Let r_i^{G} be the ith residual of* GMRES *applied to* (1.2). *Then,*

$$\|r_i^{\mathrm{G}}\| = \min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \|\phi(b_\theta) - (\mathbf{M}^2 - |\kappa|^2\mathbf{I})p_1(\mathbf{M}^2)\phi(b_\theta) - (\mathbf{M}^2 - |\kappa|^2\mathbf{I})p_2(\mathbf{M}^2)\phi(\mathrm{i}b_\theta)\|.$$

(2) *Let r_i be the ith residual of* RL-GMRES *applied to* (1.1). *Then,*

$$\|r_i\| = \min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \|\phi(b_\theta) - (\mathbf{M} + |\kappa|\mathbf{I})p_1(\mathbf{M})\phi(b_\theta) - (\mathbf{M} + |\kappa|\mathbf{I})p_2(\mathbf{M})\phi(\mathrm{i}b_\theta)\|.$$

Proof. (1) Let $C = |\kappa|^2 I - M\overline{M}$, $\mathbf{S}$ be the matrix in (2.10) and $\mathbf{C} = \begin{bmatrix} \overline{C} & 0 \\ 0 & C \end{bmatrix}$. Then,

$$\begin{aligned}
\|r_i^{\mathrm{G}}\| &= \min_{p_{i-1}\in\mathbb{P}_{i-1}} \|b - Cp_{i-1}(C)b\| = \min_{p_{i-1}\in\mathbb{P}_{i-1}} \|b_\theta - Cp_{i-1}(C)b_\theta\| \\
&= \min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \|b_\theta - Cp_1(C)b_\theta - \mathrm{i}Cp_2(C)b_\theta\| \\
&= \frac{\sqrt{2}}{2}\min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \left\| \begin{bmatrix} \overline{b_\theta} - \overline{C}p_1(\overline{C})\overline{b_\theta} + \mathrm{i}\overline{C}p_2(\overline{C})\overline{b_\theta} \\ b_\theta - Cp_1(C)b_\theta - \mathrm{i}Cp_2(C)b_\theta \end{bmatrix} \right\| \\
&= \frac{\sqrt{2}}{2}\min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \left\| \begin{bmatrix} \overline{b_\theta} - \overline{C}p_1(\overline{C})\overline{b_\theta} + \mathrm{i}\overline{C}p_2(\overline{C})\overline{b_\theta} \\ -\mathrm{i}b_\theta - Cp_1(C)(-\mathrm{i}b_\theta) - \mathrm{i}Cp_2(C)(-\mathrm{i}b_\theta) \end{bmatrix} \right\| \\
&= \frac{\sqrt{2}}{2}\min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \left\| \begin{bmatrix} \overline{b_\theta} \\ -\mathrm{i}b_\theta \end{bmatrix} - \begin{bmatrix} \overline{C}p_1(\overline{C})\overline{b_\theta} \\ Cp_1(C)(-\mathrm{i}b_\theta) \end{bmatrix} + \begin{bmatrix} \overline{C}p_2(\overline{C})\mathrm{i}\overline{b_\theta} \\ Cp_2(C)(-b_\theta) \end{bmatrix} \right\| \\
&= \min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \|\mathbf{S}\phi(b_\theta) - \mathbf{C}p_1(\mathbf{C})\mathbf{S}\phi(b_\theta) - \mathbf{C}p_2(\mathbf{C})\mathbf{S}\phi(\mathrm{i}b_\theta)\| \\
&= \min_{p_1,p_2\in\mathbb{P}_{i-1}^{\mathbb{R}}} \|\phi(b_\theta) - (\mathbf{M}^2 - |\kappa|^2\mathbf{I})p_1(\mathbf{M}^2)\phi(b_\theta) - (\mathbf{M}^2 - |\kappa|^2\mathbf{I})p_2(\mathbf{M}^2)\phi(\mathrm{i}b_\theta)\|.
\end{aligned}$$

The last equality holds due to (2.11)-(2.12), Lemma 3.5 and $\mathbf{M}_2^2 = \mathbf{M}^2$.

(2) Note that $\mathbf{M} + |\kappa|\mathbf{I}$ is the coefficient matrix of the R2 formulation of the following $\mathbb{R}$-linear system

$$|\kappa|z + e^{-\mathrm{i}\theta}M\bar{z} = e^{-\mathrm{i}\theta}b.$$

Then the result follows from Theorem 3.4 (1), Proposition 3.2, Lemma 3.5, $\mathbf{JM} = -\mathbf{MJ}$ and $\mathbf{J}\phi(b_\theta) = \phi(\mathrm{i}b_\theta)$. ◻

For the case of $\kappa = 0$, (1.1) and (1.2) reduce to

$$M\bar{z} = b, \tag{3.9}$$

and

$$-M\overline{M}w = b, \tag{3.10}$$

respectively. By Proposition 3.6, we immediately obtain the following corollary.

Corollary 3.7. *Let r_i and r_i^{G} be the* ith *residual of* RL-GMRES *applied to* (3.9) *and the* ith *residual of* GMRES *applied to* (3.10)*, respectively. Then,* $\|r_{2i}\| \le \|r_i^{\mathrm{G}}\|$.

Actually, the inequality $\|r_{2i}\| \le \|r_i^{\mathrm{G}}\|$ holds for any $\kappa \in \mathbb{C}$. To prove this, we need the following lemma.

Lemma 3.8. *Let* $\mathbf{M}$ *and* $\mathbf{I}$ *be the matrices in* Proposition 3.6*. Then, for any* $p \in \mathbb{P}_{i-1}^{\mathbb{R}}$*, there exists a polynomial* $\tilde{p} \in \mathbb{P}_{2i-1}^{\mathbb{R}}$ *satisfying*

$$(\mathbf{M} + |\kappa|\mathbf{I})\tilde{p}(\mathbf{M}) = (\mathbf{M}^2 - |\kappa|^2\mathbf{I})p(\mathbf{M}^2).$$

Proof. Note that for any $p \in \mathbb{P}_{i-1}^{\mathbb{R}}$, there exists a polynomial $\tilde{p} \in \mathbb{P}_{2i-1}^{\mathbb{R}}$ such that

$$\tilde{p}(\mathbf{M}) = (\mathbf{M} - |\kappa|\mathbf{I})p(\mathbf{M}^2). \tag{3.11}$$

Multiplying both sides of (3.11) by $\mathbf{M} + |\kappa|\mathbf{I}$, we complete the proof. ◻

Theorem 3.9. *Let r_i and r_i^{G} be the* ith *residual of* RL-GMRES *applied to* (1.1) *and the* ith *residual of* GMRES *applied to* (1.2)*, respectively. Then,*

$$\|r_{2i}\| \le \|r_i^{\mathrm{G}}\|.$$

Proof. This is a direct result of Proposition 3.6 and Lemma 3.8. ◻

Remark 3.10. By Theorem 3.9, RL-GMRES applied to (1.1) requires fewer matrix-vector products than GMRES applied to (1.2) (Note that GMRES applied to (1.2) requires two matrix-vector products every iteration, and for z an extra matrix-vector product is required). See the numerical experiments in section 4.

Remark 3.11. We can prove Theorem 3.9 through a different approach by the shift-invariance property of Krylov subspaces and the minimal residual property of RL-GMRES and GMRES; see [5, 15].

4. Numerical experiments. In this section, we report numerical results of two examples. Throughout, the computation is performed in MATLAB 2008a on a laptop with 2.26G CPU and 4GB memory.

4.1. Example 1. The matrix M used in this example is a randomly chosen complex tridiagonal matrix of dimension $n = 200$. The right-hand side b is a randomly chosen complex vector. The matrix M and the right-hand side b are generated by the following MATLAB codes.

```
n = 200 ;
d1 = rand(n,1)+i*rand(n,1);
d2 = rand(n-1,1)+i*rand(n-1,1);
d3 = rand(n-1,1)+i*rand(n-1,1);
M = diag(d1)+diag(d2,-1)+diag(d3,1);
b = rand(n,1)+i*rand(n,1);
```

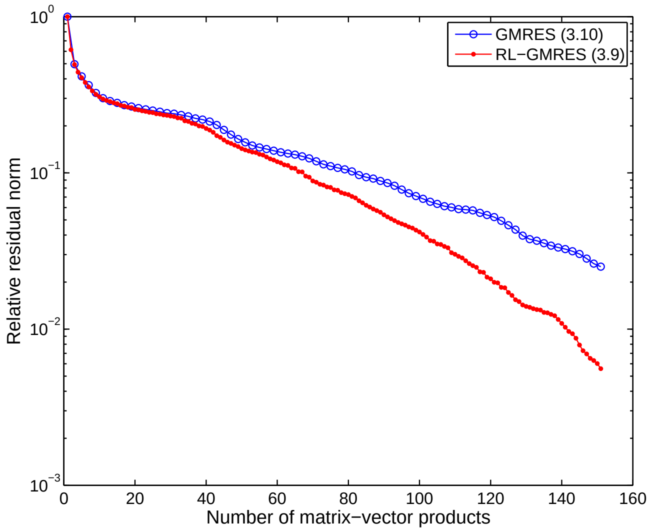

Fig. 4.1. *Example* 1. *Convergence history for* GMRES *applied to* (3.10) *and* RL-GMRES *applied to* (3.9).

We compare RL-GMRES applied to (3.9) with GMRES applied to (3.10). The initial guess is set to be the zero vector and the iteration stops if the number of matrix-vector products is 150. Note that GMRES applied to (3.10) requires two matrix-vector products every iteration and RL-GMRES requires only one. We have tested this example many times and Figure 4.1 shows the typical convergence history of these methods. The curve of RL-GMRES applied to (3.9) is below the curve of GMRES applied to (3.10), which illustrates Corollary 3.7.

4.2. Example 2. In this example, we consider an $\mathbb{R}$-linear system arising from the inverse problem of reconstructing an unknown electric conductivity [1, 11]. More precisely, we need to solve the $\mathbb{R}$-linear integral equation

$$u + (\bar{\nu}_1 \mathcal{I}_1 + \bar{\nu}_2 \mathcal{I}_2)\bar{u} = \bar{\nu}_3, \tag{4.1}$$

defined in the rectangle $\Omega = [-1, 1)^2$. The integral operators $\mathcal{I}_1$ and $\mathcal{I}_2$ are defined as follows, for $f \in C_0^\infty(\mathbb{C})$,

$$\mathcal{I}_1 f(\xi) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\zeta-\xi|>\varepsilon} \frac{f(\zeta)}{(\zeta-\xi)^2} \mathrm{d}\zeta_1 \mathrm{d}\zeta_2, \quad \mathcal{I}_2 f(\xi) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - \xi} \mathrm{d}\zeta_1 \mathrm{d}\zeta_2,$$

where $\zeta = \zeta_1 + \mathrm{i}\zeta_2$. We remark that $\mathcal{I}_1$ (Beurling transform) is a singular integral operator and $\mathcal{I}_2$ (Cauchy transform) is a weakly singular integral operator.

The numerical discretization of (4.1) on uniform $n \times n$ grids results in $\mathbb{R}$-linear systems of the form

$$z + (D_1 T_1 + D_2 T_2)\bar{z} = b, \tag{4.2}$$

where D_1 and D_2 are diagonal matrices, and T_1 and T_2 are block-Toeplitz-Toeplitz-block (BTTB) matrices. More precisely, T_1 is complex symmetric and has the structure

$$T_1 = -\frac{1}{\pi}\begin{bmatrix} M_1 & M_2 & \cdots & M_{n-1} & M_n \\ M_2^T & M_1 & M_2 & \cdots & M_{n-1} \\ \vdots & M_2^T & M_1 & \ddots & \vdots \\ M_{n-1}^T & \cdots & \ddots & \ddots & M_2 \\ M_n^T & M_{n-1}^T & \cdots & M_2^T & M_1 \end{bmatrix},$$

where $M_k = (m_{ij}^k)$, $i, j, k = 1, \ldots, n$ are Toeplitz matrices with

$$m_{ij}^k = \begin{cases} \dfrac{1}{(j-i+(k-1)\mathrm{i})^2}, & k \neq 1 \text{ or } i \neq j, \\ 0, & k = 1 \text{ and } i = j. \end{cases}$$

The matrix T_2 is complex skew-symmetric and has the structure

$$T_2 = -\frac{h}{\pi}\begin{bmatrix} N_1 & N_2 & \cdots & N_{n-1} & N_n \\ -N_2^T & N_1 & N_2 & \cdots & N_{n-1} \\ \vdots & -N_2^T & N_1 & \ddots & \vdots \\ -N_{n-1}^T & \cdots & \ddots & \ddots & N_2 \\ -N_n^T & -N_{n-1}^T & \cdots & -N_2^T & N_1 \end{bmatrix},$$

where $h = 2/n$, $N_k = (n_{ij}^k)$, $i, j, k = 1, \ldots, n$ are Toeplitz matrices with

$$n_{ij}^k = \begin{cases} \dfrac{1}{j-i+(k-1)\mathrm{i}}, & k \neq 1 \text{ or } i \neq j, \\ 0, & k = 1 \text{ and } i = j. \end{cases}$$

Let $\nu_1(\zeta) = -\exp(-\mathrm{i}(k\zeta + \bar{k}\bar{\zeta}))\dfrac{1-\varphi(\zeta)}{1+\varphi(\zeta)}$, $\nu_2(\zeta) = -\mathrm{i}\bar{k}\nu_1(\zeta)$, $\nu_3(\zeta) = -\nu_2(\zeta)$, where $k \in \mathbb{C}$ is a parameter, and the piecewise continuous conductivity $\varphi(\zeta)$ in the unit square is

$$\varphi(\zeta) = \begin{cases} 3, & |\zeta + 0.3\mathrm{i}| < 0.3, \\ 0.3, & |\zeta + 0.4 - 0.3\mathrm{i}| < 0.3 \text{ or } |\zeta - 0.4 - 0.3\mathrm{i}| < 0.3, \\ 1, & \text{otherwise.} \end{cases}$$

We have $M = D_1 T = D_1(T_1 + \mathrm{i}kT_2)$. The multiplication by M consists of a multiplication by the BTTB matrix T followed by a diagonal matrix D_1, which can be obtained in $\mathcal{O}(n^2 \log n)$ operations by FFT. We compare the performance of RL-GMRES applied to (1.1) with GMRES applied to (1.2). The initial guess is set to be the zero vector and the iteration stops if $\|b - z_i - M\bar{z}_i\|/\|b\| \leq 10^{-12}$ (RL-GMRES) or $\|b - |\kappa|^2 w_i + M\overline{M}w_i\|/\|b\| \leq 10^{-12}$ (GMRES).

We plot in Figure 4.2 the eigenvalues of the matrices $I - M\overline{M}$ and $\mathbf{R}_2$ arising from the numerical discretization on a 64×64 grid. The spectral radius of $M\overline{M}$, $\rho(M\overline{M}) \approx 0.3509 < 1$. It is obvious that $\sigma(I - M\overline{M})$ is located in the right-half plane. By Theorem 2.3, $\sigma(\mathbf{R}_2)$ is located in the right-half plane. See Figure 4.2. Table 4.1 shows the numbers of matrix-vector products of RL-GMRES applied to (1.1) and GMRES applied to (1.2) for numerical discretizations of (4.1) on uniform $N \times N$ grids. We observe that both RL-GMRES and GMRES converge fast and RL-GMRES is slightly faster than GMRES in terms of matrix-vector products.

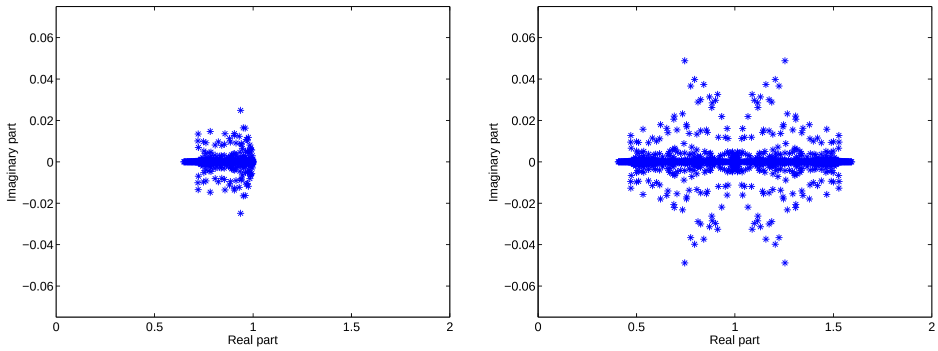

FIG. 4.2. *Example* 2. *Eigenvalues of* $I - M\overline{M}$ (*left*) *and* $\mathbf{R}_2$ (*right*)

TABLE 4.1
Example 2. *Number of matrix-vector products of* RL-GMRES *applied to* (1.1) *and* GMRES *applied to* (1.2) *for numerical discretizations of* (4.1) *on uniform* $n \times n$ *grids.*

Method	64×64	128×128	256×256	512×512
RL-GMRES	26	24	24	23
GMRES	27	27	25	25

5. Conclusions. We have analyzed $\mathbb{R}$-linear GMRES for (1.1) through the equivalent real formulation (2.2). We have proved that RL-GMRES applied to the $\mathbb{R}$-linear system (1.1) is faster than GMRES applied to the related $\mathbb{C}$-linear system (1.2) in terms of matrix-vector products. For many challenging $\mathbb{R}$-linear systems such that $\|M\| \gg |\kappa|$ (see [1]), efficient preconditioning techniques are being considered.

Acknowledgments. The authors thank Marko Huhtanen and Allan Perämäki for many helpful discussions about this work and providing their RL-GMRES MATLAB codes. Kui Du also thanks Jianfeng Cai, Graeme Fairweather, Weiwei Sun, and Yimin Wei for advice on many topics.

REFERENCES

[1] K. ASTALA, J. MUELLER, L. PÄIVÄRINTA, AND S. SILTANEN, *Numerical computation of complex geometrical optics solutions to the conductivity equation*, Appl. Comp. Harm. Anal., to appear (2010).
[2] O. AXELSSON AND A. KUCHEROV, *Real valued iterative methods for solving complex symmetric linear systems*, Numer. Linear Algebra Appl., 7 (2000), pp. 197–218.
[3] M. BENZI AND D. BERTACCINI, *Block preconditioning of real-valued iterative algorithms for complex linear systems*, IMA J. Numer. Anal., 28 (2008), pp. 598–618.
[4] D. DAY AND M. A. HEROUX, *Solving complex-valued linear systems via equivalent real formulations*, SIAM J. Sci. Comput., 23 (2001), pp. 480–498.
[5] K. DU, *Global $\mathbb{R}$-linear GMRES for solving a class of $\mathbb{R}$-linear matrix equations*, in preparation, (2010).
[6] T. EIROLA, M. HUHTANEN, AND J. VON PFALER, *Solution methods for $\mathbb{R}$-linear problems in $\mathbb{C}^n$*, SIAM J. Matrix Anal. Appl., 25 (2004), pp. 804–828.
[7] R. W. FREUND, *Conjugate gradient-type methods for linear systems with complex symmetric coefficient matrices*, SIAM J. Sci. Statist. Comput., 13 (1992), pp. 425–448.
[8] R. A. HORN AND C. R. JOHNSON, *Matrix analysis*, Cambridge University Press, 1990.
[9] M. HUHTANEN AND O. NEVANLINNA, *Real linear matrix analysis*, Banach Center Publ., 75 (2007), pp. 171–189.
[10] M. HUHTANEN AND O. NEVANLINNA, *The real linear resolvent and cosolvent operators*, J. Operator Theory, 58 (2007), pp. 229–250.

[11] M. Huhtanen and A. Perämäki, *Numerical solution of the $\mathbb{R}$-linear Beltrami equation*, submitted, (2010).
[12] M. Huhtanen and S. Ruotsalainen, *Real linear operator theory and its applications*, submitted, (2010).
[13] K. Knudsen, J. Mueller, and S. Siltanen, *Numerical solution method for the dbar-equation in the plane*, J. Comput. Phys., 198 (2004), pp. 500–517.
[14] J. Mueller and S. Siltanen, *Direct reconstructions of conductivities from boundary measurements*, SIAM J. Sci. Comput., 24 (2003), pp. 1232–1266.
[15] A. Perämäki, Institute of Mathematics, Aalto University, Personal Communication, (2010).
[16] Y. Saad, *Iterative Methods for Sparse Linear Systems*, SIAM, Philadelphia, second ed., 2003.
[17] Y. Saad and M. H. Schultz, *GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems*, SIAM J. Sci. Statist. Comput., 7 (1986), pp. 856–869.
[18] S. Siltanen, J. Mueller, and D. Isaacson, *An implementation of the reconstruction algorithm of A. Nachman for the 2D inverse conductivity problem*, Inverse Problems, 16 (2000), pp. 681–699.
[19] V. Simoncini and D. B. Szyld, *Recent computational developments in Krylov subspace methods for linear systems*, Numer. Linear Algebra Appl., 14 (2007), pp. 1–59.
[20] L. N. Trefethen and D. Bau, *Numerical linear algebra*, SIAM, 1997.
[21] H. A. van der Vorst, *Iterative Krylov Methods for Large Linear Systems*, Cambridge University Press, Cambridge, 2003.

(continued from the back cover)

A580 Juho Könnö, Dominik Schötzau, Rolf Stenberg
Mixed finite element methods for problems with Robin boundary conditions
November 2009

A579 Lasse Leskelä, Falk Unger
Stability of a spatial polling system with greedy myopic service
September 2009

A578 Jarno Talponen
Special symmetries of Banach spaces isomorphic to Hilbert spaces
September 2009

A577 Fernando Rambla-Barreno, Jarno Talponen
Uniformly convex-transitive function spaces
September 2009

A576 S. Ponnusamy, Antti Rasila
On zeros and boundary behavior of bounded harmonic functions
August 2009

A575 Harri Hakula, Antti Rasila, Matti Vuorinen
On moduli of rings and quadrilaterals: algorithms and experiments
August 2009

A574 Lasse Leskelä, Philippe Robert, Florian Simatos
Stability properties of linear file-sharing networks
July 2009

A573 Mika Juntunen
Finite element methods for parameter dependent problems
June 2009

A572 Bogdan Bojarski
Differentiation of measurable functions and Whitney–Luzin type structure theorems
June 2009

HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS
RESEARCH REPORTS

The reports are available at *http://math.tkk.fi/reports/* .
The list of reports is continued inside the back cover.

A585 Daniel Aalto
Boundedness of maximal operators and oscillation of functions in metric measure spaces
March 2010

A584 Tapio Helin
Discretization and Bayesian modeling in inverse problems and imaging
February 2010

A583 Wolfgang Desch, Stig-Olof Londen
Semilinear stochastic integral equations in L_p
December 2009

A582 Juho Könnö, Rolf Stenberg
Analysis of H(div)-conforming finite elements for the Brinkman problem
January 2010

A581 Wolfgang Desch, Stig-Olof Londen
An L_p-theory for stochastic integral equations
November 2009

ISBN 978-952-60-3181-1 (print)
ISBN 978-952-60-3182-8 (PDF)
ISSN 0784-3143 (print)
ISSN 1797-5867 (PDF)