UNIFORMLY CONVEX-TRANSITIVE FUNCTION SPACES

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**Abstract:** We introduce a property of Banach spaces, called uniform convex-transitivity, which falls between almost transitivity and convex-transitivity. We will provide examples of uniformly convex-transitive spaces. This property behaves nicely in connection with some vector-valued function spaces. As a consequence, we obtain some new examples of convex-transitive Banach spaces.

*(To appear in Quarterly Journal of Mathematics)*

**AMS subject classifications:** Primary 46B04, 46B20; Secondary 46B25

**Keywords:** Banach spaces, uniform convex-transitivity, convex-transitive, almost transitive, vector-valued function spaces, rotations

**Correspondence**

Jarno Talponen  
Helsinki University of Technology  
Department of Mathematics and Systems Analysis  
P.O. Box 1100  
FI-02015 TKK  
Finland

fernando.rambla@uca.es, talponen@cc.hut.fi

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ISSN 0784-3143 (print)  
ISSN 1797-5867 (PDF)

Helsinki University of Technology  
Faculty of Information and Natural Sciences  
Department of Mathematics and Systems Analysis  
P.O. Box 1100, FI-02015 TKK, Finland  
email: math@tkk.fi http://math.tkk.fi/
1 Introduction

In this paper we study the symmetries of some well-known, in fact, almost classical Banach spaces. We denote the closed unit ball of a Banach space $X$ by $B_X$ and the unit sphere of $X$ by $S_X$. A Banach space $X$ is called transitive if for each $x \in S_X$ the orbit $\mathcal{G}_X(x) = \{T(x) | T: X \to X$ is an isometric automorphism\} = $S_X$. If $\cap \mathcal{G}_X(x) = S_X$ (resp. $\cap \mathcal{G}_X(x) \cap B_X$) for all $x \in S_X$, then $X$ is called almost transitive (resp. convex-transitive).

These concepts are motivated by the Banach-Mazur rotation problem appearing in [2, p.242], which remains unsolved. We refer to [5] and [8] for a survey and discussion on the matter.

The known concrete examples of convex-transitive spaces are scarce, and the ultimate aim of this paper is to provide more examples by establishing the convex-transitivity of some vector-valued function spaces and other natural spaces. It was first reported by Pelczynski and Rolewicz [18] in 1962 that the space $L^p$ is almost transitive for $p \in [1, \infty)$ and convex-transitive for $p = \infty$ (see also [20]). Later, Wood [23] characterized the spaces $C_0^R(L)$ whose norm is convex-transitive (see Preliminaries). Greim, Jamison and Kamin ska [14] proved that if $X$ is almost transitive and $1 \leq p < \infty$, then the Lebesgue-Bochner space $L^p(X)$ is also almost transitive. Recently, an analogous study of the spaces $C_0^R(L, X)$ was done by Aizpuru and Rambla [1], and some related spaces were studied by Talponen [22]. For some other relevant contemporary results, see [7], [16] and [19].

We will extend these investigations into the vector-valued convex-transitive setting, which differs considerably in many respects from the scalar-valued almost transitive one. For this purpose we will introduce a new concept which is (formally) stronger than convex-transitivity and weaker than almost transitivity, called uniform convex-transitivity. With the aid of this class of Banach spaces we produce new natural examples of convex-transitive vector-valued function spaces. The main results of this paper are the following:

- Characterization of locally compact Hausdorff spaces $L$ such that $C_0^R(L)$ is uniformly convex-transitive.
- If $X$ is a uniformly convex-transitive Banach space, then so is $L_{R_0}^\infty(X)$.
- If $X$ and $C_0^R(L)$ are uniformly convex-transitive, then so is $C_0^R(L, X)$.

Preliminaries

The scalar field of a Banach space $X$ is denoted by $\mathbb{K}$ and whenever there are several Banach spaces under discussion, then $\mathbb{K}$ is the scalar field of the space denoted by $X$. The open unit ball of $X$ is denoted by $U_X$. The group of rotations $\mathcal{G}_X$ of $X$ consists of isometric automorphisms $T: X \to X$, the group operation being the composition of the maps and the neutral element being the identity map $I: X \to X$. We will always consider $\mathcal{G}_X$ equipped with the strong operator topology (SOT). An element $x \in S_X$ is called a big point if
\text{conv}\mathcal{G}(x) = B_X. \text{ Thus } X \text{ is convex-transitive if and only if each } x \in S_X \text{ is a big point.}

Recall that a topological space is totally disconnected if each connected component of the space is a singleton. In what follows \( L \) is a locally compact Hausdorff space and \( K \) is a compact Hausdorff space, unless otherwise stated. In [23] Wood characterized convex-transitive \( C^R_0(L) \) spaces. Namely, \( C^R_0(L) \) is convex-transitive if and only if \( L \) is totally disconnected and for every regular probability measure \( \mu \) on \( L \) and \( t \in L \) there exists a net \( \{ \gamma_\alpha \}_\alpha \) of homeomorphisms on \( L \) such that the net \( \{ \mu \circ \gamma_\alpha \}_\alpha \) is \( \omega^* \)-convergent to the Dirac measure \( \delta_t \). The above mapping \( \mu \circ \gamma_\alpha \) is given by \( \mu \circ \gamma_\alpha(A) = \mu(\gamma_\alpha(A)) \) for Borel sets \( A \subset L \).

We refer to [17] for background information on measure algebras and isometries of \( L^p \)-spaces and to [11] for a suitable source to Banach spaces in general. In what follows \( \Sigma \) is the completed \( \sigma \)-algebra of Lebesgue measurable sets on \([0, 1]\) and we denote by \( m: \Sigma \to \mathbb{R} \) the Lebesgue measure. Define an equivalence relation \( m \sim \) on \( \Sigma \) by setting \( A \sim B \) if \( m((A \cup B) \setminus (A \cap B)) = 0 \).

Recall that a rotation \( R \) on the space \( C^K_0(L, X) \) is said to be of the Banach-Stone type, if \( R \) can be written as

\[
R(f)(t) = \sigma(t)(f \circ \phi(t)), \quad f \in C^K_0(L, X),
\]

where \( \phi: L \to L \) is a homeomorphism and \( \sigma: L \to \mathcal{G}_X \) is a continuous map.

A Banach space \( Y \) is said to be contained \textit{almost isometrically} in a Banach space \( X \) if for each \( \varepsilon > 0 \) there is a linear map \( \psi: Y \to X \) such that

\[
||y||_Y \leq ||\psi(y)||_X \leq (1 + \varepsilon)||y||_Y \quad \text{for } y \in Y.
\]

**2 Uniform convex-transitivity**

Provided that the space \( X \) under discussion is understood, we denote

\[
C_n(x) = \left\{ \sum_{i=1}^n a_i T_i(x) \mid T_1, \ldots, T_n \in \mathcal{G}_X, \ a_1, \ldots, a_n \in [0, 1], \ \sum_{i=1}^n a_i = 1 \right\}
\]

for \( n \in \mathbb{N} \) and \( x \in S_X \). We call a Banach space \( X \) \textit{uniformly convex-transitive} if for each \( \varepsilon > 0 \) there exists \( n \in \mathbb{N} \) satisfying the following condition: For all \( x \in S_X \) and \( y \in B_X \) it holds that \( \text{dist}(y, C_n(x)) \leq \varepsilon \), that is

\[
\lim_{n \to \infty} \sup_{x \in S_X, y \in B_X} \text{dist}(y, C_n(x)) = 0.
\]

We denote by \( K_\varepsilon \) the least integer \( n \), which satisfies the above inequality involving \( \varepsilon \) and such \( K_\varepsilon \) is called \textit{the constant of uniform convex transitivity of } \( X \text{ associated to } \varepsilon \). We call \( x \in S_X \) a \textit{uniformly big point} if

\[
\lim_{n \to \infty} \sup_{y \in B_X} \text{dist}(y, C_n(x)) = 0.
\]
Clearly almost transitive spaces are uniformly convex-transitive, and uniformly convex-transitive spaces are convex-transitive. It is well-known that $C^0(S^1)$ is a convex-transitive, non-almost transitive space, and it is easy to see (see e.g. the subsequent Theorem 2.4) that it is even uniformly convex-transitive. Unfortunately, we have not been able so far to find an example of a convex-transitive space which is not uniformly convex-transitive. However, we suspect that such examples exist and we note that the absence of such a complicated space would make some proofs regarding convex-transitive spaces much more simple. Observe that the canonical unit vectors $e_k \in \ell^1$ are far from being uniformly big points:

$$\lim_{n \to \infty} \sup_{y \in B_{\ell^1}} \text{dist}(y, C_n(e_k)) = 1,$$

even though they are big points, i.e. $\overline{\text{conv}}(G_{\ell^1}(e_k)) = B_{\ell^1}$ for $k \in \mathbb{N}$. In any case, we will provide examples of uniformly convex-transitive spaces, most of which are not previously known to be even convex-transitive.

We note that if $X$ is convex-transitive and there exists a uniformly big point $x \in S_X$, then each $y \in S_X$ is a uniformly big point. This does not mean, a priori, that $X$ should be uniformly convex-transitive. Next we give an equivalent condition to uniform convex transitivity, which is more applicable in calculations than the condition introduced above.

**Proposition 2.1.** Let $X$ be a Banach space. The following condition of $X$ is equivalent to $X$ being uniformly convex-transitive: For each $\varepsilon > 0$ there is $N_{\varepsilon} \in \mathbb{N}$ such that for each $x \in S_X$ and $y \in B_X$ there are $T_1, \ldots, T_{N_{\varepsilon}} \in G_X$ such that

$$\left\| y - \frac{1}{N_{\varepsilon}} \sum_{i=1}^{N_{\varepsilon}} T_i(x) \right\| \leq \varepsilon. \quad (1)$$

**Proof.** It is clear that (1) implies uniform convex transitivity, even for the value $K_{\varepsilon} = N_{\varepsilon}$ for each $\varepsilon > 0$. Towards the other direction, let $X$ be a uniformly convex-transitive Banach space, $\varepsilon > 0$ and $x \in S_X$, $y \in B_X$. Let $K$ be the constant of uniform convex-transitivity of $X$ associated to $\frac{\varepsilon}{4}$. Then there are $a_1, \ldots, a_K \in [0, 1]$, $\sum_i a_i = 1$ and $T_1, \ldots, T_K \in G_X$ such that

$$\left\| y - \sum_{i=1}^{K} a_i T_i(x) \right\| \leq \frac{\varepsilon}{4}.$$

Put $m = \lceil \frac{4K}{\varepsilon} \rceil \in \mathbb{N}$, so that $K \cdot \frac{1}{m} \leq \frac{\varepsilon}{4}$. Next we define an $m$-uple $(S_1, \ldots, S_m) \subset G_X$ as follows: For each $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, K\}$ we put $S_j = T_i$ if $[m \sum_{n<i} a_n] < j \leq [m \sum_{n\leq i} a_n]$. (Here $\sum_{\emptyset} a_n = 0$.) By applying the triangle inequality several times, we obtain that

$$\left\| y - \frac{1}{m} \sum_{j=1}^{m} S_j(x) \right\| \leq \varepsilon.$$

Hence it suffices to put $N_{\varepsilon} = m = \lceil \frac{4K}{\varepsilon} \rceil$, where $K$ depends only on the Banach space $X$ and the value of $\varepsilon$. □
In what follows, we will apply the constant $N_\varepsilon$ freely without explicit reference to the previous proposition, and if there is no danger of confusion, also without mentioning explicitly $X$ and $\varepsilon$, either.

The following condition on the locally compact space $L$ turns out to be closely related to the uniform convex-transitivity of $C^g_0(L)$:

(*) For each $\varepsilon > 0$ there is $M_\varepsilon \in \mathbb{N}$ such that for every non-empty open subset $U \subset L$ and compact $K \subset L$ there are homeomorphisms $\phi_1, \ldots, \phi_{M_\varepsilon} : L \to L$ with

$$\frac{1}{M_\varepsilon} \sum_{i=1}^{M_\varepsilon} \chi_{\phi_i^{-1}(U)}(t) \geq 1 - \varepsilon \quad \text{for } t \in K.$$ 

This condition should be compared with the conditions found by Cabello (see [7, p.110-113], especially condition (g)), which characterize the convex transitivity of $C_0(L)$. Next we will give this characterization the uniformly convex-transitive counterpart. If $L$ is a locally compact Hausdorff space, by $\alpha L$ we denote its one-point compactification and if $L$ is noncompact, we denote such point by $\infty$. Prior to the theorem we need the following two lemmas.

**Lemma 2.2.** ([19, Thm. 3.1]) Let $T$ be a normal topological space with $\dim T \leq 1$. If $F \subseteq T$ is a closed subset and $f : F \to S_\mathbb{C}$ is a continuous map, then $f$ admits a continuous extension $g : T \to S_\mathbb{C}$.

**Lemma 2.3.** Let $L$ be a locally compact, Hausdorff, 0-dimensional space. Then for every $g \in \mathcal{B}_{C^0_0(L)}$ and $k \in \mathbb{N}$ there exist disjoint clopen sets $C_1, C_2, \ldots, C_{2k-1}$ such that the function $h \in \mathcal{B}_{C^0_0(L)}$ defined by $h = \sum_{i=1}^{2k-1} \frac{i-k}{k} \chi_{C_i}$ satisfies $\|h - g\| \leq \frac{3}{2k}$.

**Proof.** We regard $g$ as defined in $\alpha L$. Consider $i \in \{-k, -k+1, \ldots, k-1\}$ and let $K_i = g^{-1}[\frac{i}{k}, \frac{i+1}{k}]$. Every $x \in K_i$ has a clopen neighbourhood $A_x$ such that $g(A_x) \subseteq [\frac{2i-1}{2k}, \frac{2i+3}{2k}]$. By compactness there exist $x_1, \ldots, x_n$ such that $K_i \subseteq \bigcup_{j=1}^n A_{x_j} = B_i$. Finally, define $C_0 = B_0$, $C_1 = B_{-k} \setminus C_0$, $\ldots$, $C_k = B_{-1} \setminus (C_0 \cup \cdots \cup C_{k-1})$, $C_{k+1} = B_1 \setminus (C_0 \cup \cdots \cup C_k)$, $\ldots$, $C_{2k-1} = B_{k-1} \setminus (C_0 \cup \cdots \cup C_{2k-2})$. Note that the $C_i$’s are a partition of $\alpha L$.

Now take $h : L \to \mathbb{R}$ given by $h = \sum_{i=1}^{2k-1} \frac{i-k}{k} \chi_{C_i}$. It is easy to check that $\|h - g\| \leq \frac{3}{2k}$ and $h \in \mathcal{B}_{C^0_0(L)}$. \qed

**Theorem 2.4.** Let $L$ be a locally compact Hausdorff space. The space $C^g_0(L)$ is uniformly convex-transitive if and only if $L$ is totally disconnected and satisfies (*). If the space $C^c_0(L)$ is uniformly convex-transitive, then $L$ satisfies (*). Moreover, if $\dim(\alpha L) \leq 1$, then also the converse implication holds.

Before the proof we comment on the above assumptions.

**Remark 2.5.** The spaces $C^r(S^1, \mathbb{R}^2)$ and $C^c(S^1, \mathbb{C})$ are uniformly convex-transitive, their rotations are of the Banach-Stone type, and clearly $S^1$, $\mathcal{G}_{\mathbb{R}^2}$ and $\mathcal{G}_\mathbb{C}$ are not totally disconnected.
Proof of Theorem 2.4. Let us first consider the only if directions. Since uniformly convex-transitive spaces are convex-transitive, we may apply Wood’s characterization for convex-transitive \( C^R_0(L) \) spaces, and thus we obtain that \( L \) must be totally disconnected. Let \( C^R_0(L), \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\} \), be uniformly convex-transitive. Next we aim to check that \( L \) satisfies (*), so let \( U \subset L \) be a non-empty open subset and \( K \subset L \) a compact subset. Fix \( x_0 \in U \). Since \( \alpha L \) is normal, there exist continuous functions \( f, g : \alpha L \to [-1, 1] \) satisfying \( f(\alpha L \setminus U) = \{0\} \), \( f(x_0) = 1 \), \( g(K) = \{1\} \) and \( g(\infty) = 0 \). Since both functions vanish at infinity, we can consider that \( f, g \in S_{C^R_0(L)} \).

Fix \( \varepsilon > 0 \) appearing in condition (*). Let \( N_\varepsilon \) be the associated constant provided by the uniform convex-transitivity and condition (1). Then by the definition of \( N_\varepsilon \) and the Banach-Stone characterization of rotations of \( C^R_0(L) \) we obtain that there are continuous functions \( \sigma_1, \ldots, \sigma_{N_\varepsilon} : L \to \mathbb{K} \) and homeomorphisms \( \phi_1, \ldots, \phi_{N_\varepsilon} : L \to L \) such that

\[
\left\| g - \frac{1}{N_\varepsilon} \sum_{i=1}^{N_\varepsilon} \sigma_i(f \circ \phi_i) \right\| \leq \varepsilon. \tag{2}
\]

In particular, this yields for each \( t \in K \) that

\[
\varepsilon \geq |1 - \frac{1}{N_\varepsilon} \sum_{i=1}^{N_\varepsilon} \sigma_i f(\phi_i(t))| = \left| \frac{1}{N_\varepsilon} \sum_{i=1}^{N_\varepsilon} 1 - \sigma_i f(\phi_i(t)) \right|
\]

where we applied the fact that \( f \) vanishes outside \( U \). This justifies (*) for \( M_\varepsilon = N_\varepsilon \).

Let us see the if direction for \( C^R_0(L) \). Let \( k \in \mathbb{N} \), \( f \in S_{C^R_0(L)} \) and \( g \in B_{C^R_0(L)} \). We may assume max \( f = 1 \). Take \( h \) as in Lemma 2.3, i.e. \( h = \sum_{i=1}^{2k-1} \frac{i-k}{k} \chi_{C_i} \) with each \( C_i \) clopen and \( \|h - g\| \leq \frac{3}{2k} \).

Note that \( K = \bigcup_{i=1}^{2k-1} C_i \) is compact and apply (*) to this \( K \), the subset \( U = \{t \in L : f(t) > 1 - k^{-1}\} \) and \( \varepsilon = \frac{1}{k} \). Write \( M = M_\varepsilon \). There exist homeomorphisms \( \phi_1, \ldots, \phi_M \) such that if \( t \in K \) then \( \frac{1}{M} \sum_{i=1}^{M} \chi_{\phi_i^{-1}(U)}(t) \geq 1 - k^{-1} \). For each \( j \in \{1, \ldots, 2k - 1\} \), define \( B_j = \bigcup_{s=1}^{2k-1} C_s \) and let \( T_j \) be the rotation on \( C^R_0(L) \) given by \( T_j x = (\chi_{B_j} - \chi_{B_j}^c) \cdot x \) if \( j \leq k \) and \( T_j x = (\chi_{B_j} - \chi_{B_j}^c + 2\chi_{L \setminus K}) \cdot x \) if \( j > k \). Now only a few calculations are needed to see that

\[
\left\| g - \frac{1}{M(2k-1)} \sum_{j=1}^{2k-1} \sum_{i=1}^{M} T_j(f \circ \phi_i) \right\| \leq 6k^{-1}
\]

and thus \( C^R_0(L) \) is uniformly convex transitive.

In order to justify the last sentence in the theorem it is required to verify that if \( L \) satisfies \( \dim(\alpha L) \leq 1 \) and (*), then \( C^R_0(L) \) is uniformly convex-transitive. Let \( k \in \mathbb{N} \) and let \( M_k \) be the corresponding constant in condition (*) associated to value \( k^{-1} \). Fix \( f \in S_{C^R_0(L)} \) and \( g \in B_{C^R_0(L)} \). We may assume without loss of generality, possibly by multiplying \( f \) with a suitable complex number of modulus 1, that \( f(t_0) = 1 \) for a suitable \( t_0 \in L \). Let \( U = \{t \in L : |1 - f(t)| < k^{-1}\} \) and \( K = \{t \in L : |g(t)| \geq k^{-1}\} \). Let \( \phi_1, \ldots, \phi_{M_k} : L \to L \)
be homeomorphisms such that \( \frac{1}{M_k} \sum_{i=1}^{M_k} x_{\phi_i^{-1}(U)}(t) \geq 1 - k^{-1} \) for \( t \in K \). This means that the average

\[
F = \frac{1}{M_k} \sum_{i=1}^{M_k} f \circ \phi_i \in B_{C_0^\infty(L)}
\]

satisfies \( |1 - F(t)| \leq 3k^{-1} \) for each \( t \in K \).

Next we will define some auxiliary mappings. Put \( \alpha : S_C \times [0,1] \to S_C; \alpha(z,s) = -i^{2s}z \). Note that this is a continuous map, and \( \alpha(z,0) = -z \), \( \alpha(z,1) = z \) for \( z \in S_C \). Taking into account Lemma 2.2 with \( T = \alpha L \), let \( \beta_g : L \to S_C \) be a continuous extension of the function \( \frac{g(\cdot)}{|g(\cdot)|} \) defined on \( K \).

For \( j \in \{1,\ldots,k\} \) we define rotations on \( C_0^\infty(L) \) by putting \( e_{ja}(x)(t) = \beta_g(t) \cdot x(t) \) and \( e_{jb}(x)(t) = \alpha(\beta_g(t), \min(1, \max(0,k|g(t)| - j))) \cdot x(t) \). The main point above is that \( (e_{ja} + e_{jb})(F)(t) = 0 \) for \( (j,t) \in \{(j,t) \in \{1,\ldots,k\} \times L \) such that \( |g(t)| \leq \frac{k}{j} \) and \( (e_{ja} + e_{jb})(F)(t) = 2F(t)\beta_g(t) \) for \( (j,t) \in \{1,\ldots,k\} \times L \) such that \( g(t) \geq \frac{k}{j+1} \). Thus, by using (3) we obtain that

\[
\left\| |g(t)|\beta_g(t) - \frac{1}{M_k} \sum_{j=1}^{M_k} (e_{ja} + e_{jb})(F)(t) \right\| \leq 2k^{-1} \quad \text{for} \quad t \in L.
\]

Here \( ||g(\cdot) - |g(\cdot)|\beta_g(\cdot)|| \leq k^{-1} \), so that \( C_0^\infty(L) \) is uniformly convex-transitive.

Note that Theorem 2.4 yields the fact that if \( C_0^\infty(L) \) is uniformly convex-transitive, then so is \( C_0^\infty(L) \). By the above reasoning one can also see that if \( C_0^\infty(L) \) is convex-transitive and \( |L| > 1 \), then \( L \) contains no isolated points and thus it follows that each non-empty open subset of \( L \) is uncountable (see also [3, Thm. 1]). Cabello pointed out [7, Cor. 1] that locally compact spaces \( L \) having a basis of clopen sets \( C \) such that \( L \setminus C \) is homeomorphic to \( C \), have the property that \( C_0^\infty(L) \) is convex-transitive. Consequently, this provides a route to the fact that the spaces \( L^\infty, \ell^\infty/c_0 \) and \( C(\Delta) \) over \( \mathbb{R} \), where \( \Delta \) is the Cantor set, are convex-transitive. By applying Theorem 2.4 and following Cabello’s argument with slight modifications, one arrives at the conclusion that these spaces are in fact uniformly convex-transitive. When studying [7] it is helpful to observe that each occurrence of ‘basically disconnected’ in the paper must be read as totally disconnected, ([6]).

It is quite easy to verify that if \( L_1,\ldots,L_n \), where \( n \in \mathbb{N} \), are totally disconnected locally compact Hausdorff spaces satisfying (*), then so is the product \( L_1 \times \cdots \times L_n \). It follows that the space \( C_0^\infty(L_1 \times \cdots \times L_n) \) (also known as the injective tensor product \( C_0^\infty(L_1) \hat{\otimes} \cdots \hat{\otimes} C_0^\infty(L_n) \), up to isometry) is uniformly convex-transitive.

### 3 Uniform convex-transitivity of Banach-valued function spaces

With a proof similar to that of lemma 2.3, we obtain the following:
Lemma 3.1. Let $L$ be a locally compact, Hausdorff, 0-dimensional space and $X$ a Banach space over $\mathbb{K}$. Given $g \in B_{C_0^g(L,X)}$ and $j \in \mathbb{N}$, there exist nonzero $x_1, \ldots, x_n \in B_X$ and disjoint clopen sets $C_1, C_2, \ldots, C_n \subset L$ such that the function $h \in B_{C_0^g(L,X)}$ defined by $h(t) = \sum_{i=1}^n \chi_{C_i}(t)x_i$ satisfies $\|h - g\| < \frac{1}{j}$.

Theorem 3.2. Let $L$ be a locally compact Hausdorff space and $X$ a Banach space over $\mathbb{K}$. Consider the following conditions:

1. $L$ is totally disconnected and satisfies $(\ast)$, i.e. $C_0^g(L)$ is uniformly convex-transitive.

2. $X$ is uniformly convex-transitive.

3. $C_0^g(L,X)$ is uniformly convex-transitive.

We have the implication $(1) + (2) \implies (3)$. If the rotations of $C_0^g(L,X)$ are of the Banach-Stone type and $\dim_k(X) \geq 1$, then $(3) \implies (\ast) + (2)$. If additionally $\mathbb{K} = \mathbb{R}$ and $G_X$ is totally disconnected, then $(3) \implies (1) + (2)$.

Recall Remark 2.5 related to the last claim above.

Proof of Theorem 3.2. We begin by proving the implication $(1) + (2) \implies (3)$. Fix $k \in \mathbb{N}$. Then condition $(\ast)$ provides us with an integer $N_k$ associated to $\frac{1}{4k}$. Let $f \in S_{C_0^g(L,X)}$ and $g \in B_{C_0^g(L,X)}$. Take $h$ and $C_1, \ldots, C_n \subset L$ as in Lemma 3.1 with $j = 2k$.

Let $B = \bigcup_{i=1}^n C_i$ and $K = \{t \in L : \|g(t)\| \geq k^{-1}\}$. Note that $B$ is a compact clopen set and $K \subset B$. There are $y \in S_X$ and $T_1, \ldots, T_{N_k} \in G_{C_0^g(L,X)}$ such that

$$\left\| y - \left( \frac{1}{N_k} \sum_{i=1}^{N_k} T_i f \right)(t) \right\| < \frac{1}{k}, \quad \text{for } t \in B. \quad (4)$$

Indeed, observe that the continuous map $L \to \mathbb{R}; t \mapsto \|f(t)\|$ attains its supremum, the value $1$. Thus, let $t_0 \in L$ be such that $\|f(t_0)\| = 1$ and let $y = f(t_0) \in S_X$. Write $V = \{t \in L : \|f(t) - y\| < \frac{1}{2k}\}$. By using $(\ast)$ there are homeomorphisms $\sigma_1, \ldots, \sigma_{N_k} : L \to L$ such that

$$\frac{1}{N_k} \sum_{i=1}^{N_k} \chi_V(\sigma_i(t)) \geq 1 - \frac{1}{4k}, \quad \text{for } t \in B. \quad (5)$$

Let $T_i \in G_{C_0^g(L,X)}$ be given by $(T_i F)(t) = F(\sigma_i(t))$ for $1 \leq i \leq N_k$ and $F \in C_0^g(L,X)$. Thus, for all $t \in B$ we obtain by (5) and the definition of $V$ that

$$\left\| y - \left( \frac{1}{N_k} \sum_{i=1}^{N_k} T_i f \right)(t) \right\|$$

$$= \left\| \frac{1}{N_k} \sum_{i=1}^{N_k} y - \frac{1}{N_k} \sum_{i=1}^{N_k} f(\sigma_i(t)) \right\| \leq \frac{1}{N_k} \sum_{i=1}^{N_k} \|y - f(\sigma_i(t))\|$$

$$< (1 - \frac{1}{4k}) \cdot \frac{1}{2k} + \frac{1}{4k} \cdot 2 < \frac{1}{k}.$$
Since \( X \) is uniformly convex-transitive, there is an integer \( 2M = N \), satisfying (1) for the value \( \varepsilon = k^{-1} \). Let \( S_k^{(i)}, \ldots, S_{2M}^{(i)} \in \mathcal{G}_X \) for \( 1 \leq i \leq n \) such that
\[
\left\| x_i - \frac{1}{2M} \sum_{l=1}^{2M} S_l^{(i)}(y) \right\| < k^{-1} \quad \text{for } 1 \leq i \leq n. \tag{6}
\]

Then for each \( 1 \leq l \leq 2M \) we define a rotation on \( C^R_0(L, X) \) by
\[
R_l(F)(t) = \chi_{L \setminus B}(t)(-1)^l F(t) + \sum_{i=1}^n \chi_{C_i} S_i^{(l)}(F(t)), \quad F \in C^R_0(L, X), \quad t \in L.
\]

Indeed, this defines rotations, since the sets \( L \setminus B \) and \( C_i \) are clopen. It is easy to see by combining (4) and (6) that
\[
\left\| g - \frac{1}{2M} \sum_{l=1}^{2M} R_l \frac{1}{N_k} \sum_{i=1}^{N_k} T_i f \right\| < 2k^{-1}.
\]

This verifies the first implication.

Next we will prove the implication (3) \( \implies \) (5)+(2) under the assumption that the rotations are of the Banach-Stone type. In fact, the verification of claim (3) \( \iff \) (5) reduces to the analogous scalar-valued case, which was treated in the proof of Theorem 2.4. Moreover, by using the Banach-Stone representation of rotations and functions of type \( f \otimes x, g \otimes y \in S_{C^R_0(L, X)} \) it is easy to verify that the uniform convex-transitivity of \( C^R_0(L, X) \) implies that of \( X \).

Finally, let us prove the total disconnectedness of \( L \) in the case when \( \mathcal{G}_X \) is totally disconnected and \( \mathbb{K} = \mathbb{R} \). Assume to the contrary that \( L \) contains a connected subset \( C \), which is not a singleton. Pick \( t, s \in C \) with \( t \neq s \), and \( x \in \mathbb{S}_X \). Let \( x^* \in \mathbb{S}_X^* \) with \( x^*(x) = 1 \). Let \( f, g \in S_{C^R_0(L)} \) be functions with disjoint supports and such that \( f(t) = g(s) = 1 \). Consider \( f \otimes x, f \otimes x - g \otimes x \in S_{C^R_0(L, X)} \). Since \( C^R_0(L, X) \) is convex-transitive we obtain that \( f \otimes x - g \otimes x \in \text{conv}(\mathcal{G}_{C^R_0(L, X)}(f \otimes x)) \).

It follows easily by taking into account the Banach-Stone representation of rotations of \( C^R_0(L, X) \) and by studying the convex combinations in \( \text{conv}(\mathcal{G}_{C^R_0(L, X)}(f \otimes x)) \) that there exists a continuous map \( \sigma: L \to \mathcal{G}_X \) such that
\[
x^*(\sigma(t)(x)), x^*(-\sigma(s)(x)) > 0.
\]

By using the facts that \( \sigma(t) \neq \sigma(s) \) and that \( \mathcal{G}_X \) is totally disconnected we obtain that \( \sigma(C) \) is not connected. However, we have a contradiction, since \( \sigma(C) \) is a continuous image of a connected set. This contradiction shows that \( L \) must be totally disconnected. \( \square \)

By following the argument in the previous proof with slight modifications one obtains an analogous result in the convex-transitive setting.

**Theorem 3.3.** If \( C^R_0(L) \) is convex-transitive and \( X \) is a convex-transitive space over \( \mathbb{K} \), then \( C^R_0(L, X) \) is convex-transitive.
Proof. The proof of Theorem 3.2 has the convex-transitive counterpart with convex combinations of rotations in place of averages of rotations. Indeed, in the equation (4) one uses the convex-transitivity of $C^R_0(L)$ and the corresponding Banach-Stone type rotations applied on $C^K_0(L, X)$. After equation (4) the argument proceeds similarly. Note that in the convex-transitive setting there does not exist, a priori, an upper bound $M$ depending only on $\epsilon$.

Recall that the Lebesgue-Bochner space $L^p(X)$ consists of strongly measurable maps $f$: $[0, 1] \to X$ endowed with the norm

$$
||f||_{L^p(X)}^p = \int_0^1 ||f(t)||_X^p \, dt, \quad \text{for } p \in [1, \infty)
$$

and $||f||_{L^\infty(X)} = \text{ess sup}_{t \in [0,1]} ||f(t)||_X$. We refer to [10] for precise definitions and background information regarding the Banach-valued function spaces appearing here.

Recall that $L^\infty$ is convex-transitive (see [18] and [20]). Greim, Jamison and Kaminska proved that $L^p(X)$ is almost transitive if $X$ is almost transitive and $1 \leq p < \infty$, see [14, Thm. 2.1]. We will present the analogous result for uniformly convex-transitive spaces, that is, if $X$ is uniformly convex-transitive, then $L^p(X)$ are also uniformly convex-transitive for $1 \leq p \leq \infty$.

Theorem 3.4. Let $X$ be a uniformly convex-transitive space over $\mathbb{K}$. Then the Bochner space $L^p_{\mathbb{K}}(X)$ is uniformly convex-transitive for $1 \leq p \leq \infty$.

We will make some preparations before giving the proof. Suppose that $(A_n)_{n \in \mathbb{N}}$ is a countable measurable partition of the unit interval and $(x_n)_{n \in \mathbb{N}} \subset X$. We will use the short-hand notation $F = \sum_n \chi_{A_n} x_n$ for the function $F \in L^\infty(X)$ defined by $F(t) = x_n$ for a.e. $t \in A_n$ for each $n \in \mathbb{N}$. The following two auxiliary observations are obtained immediately from the fact that the countably valued functions are dense in $L^\infty(X)$ and the triangle inequality, respectively.

Fact 3.5. Consider $F = \sum_n \chi_{A_n} x_n$, where $(A_n)$ is a measurable partition of $[0, 1]$ and $(x_n) \subset B_X$. Functions $F$ of such type are dense in $B_{L^\infty(X)}$.

Fact 3.6. Let $X$ be a Banach space and $T_1, \ldots, T_n \in \mathcal{G}_X$, $n \in \mathbb{N}$. Assume that $x, y, z \in X$ satisfy $||y - \frac{1}{n} \sum_i T_i(x)|| = \varepsilon \geq 0$ and $||x - z|| = \delta \geq 0$. Then $||y - \frac{1}{n} \sum_i T_i(z)|| \leq \varepsilon + \delta$.

Proof of Theorem 3.4. We mainly concentrate on the case $p = \infty$. Fix $k \in \mathbb{N}$, $x \in S_X$, $(x_n), (y_n) \subset B_X$ and measurable partitions $(A_n)$ and $(B_n)$ of the unit interval. Let $N_\varepsilon$ be the integer provided by the uniform convex transitivity of $X$ associated to the value $\varepsilon = \frac{1}{k}$. Write

$$
F = \sum_n \chi_{A_n} x_n \quad \text{and} \quad G = \sum_n \chi_{B_n} y_n.
$$
We assume additionally that $\|F\| = 1$.

For each $n \in \mathbb{N}$ there are isometries $\{T^{(n)}_i\}_{i \leq N_k} \subset \mathcal{G}_X$ such that

$$\left\| \frac{1}{N_k} \sum_{i=1}^{N_k} T^{(n)}_i(x) - y_n \right\| < \frac{1}{k} \quad \text{for } n \in \mathbb{N}. \quad (7)$$

Observe that one obtains rotations on $L^\infty(X)$ by putting

$$R_i(f)(t) = \sum_n \chi_{B_n} T^{(n)}_i(f(t))$$

for a.e. $t \in [0, 1]$, where $f \in L^\infty(X)$, $i \leq N_k$, and the above summation is understood in the sense of pointwise convergence almost everywhere. We define a convex combination of elements of $\mathcal{G}_{L^\infty(X)}$ by

$$A_1(f) = \frac{1}{N_k} \sum_{i=1}^{N_k} R_i(f), \quad f \in L^\infty(X).$$

Condition (7) implies that

$$\|G - A_1(\chi_{[0,1]}x)\| < \frac{1}{k}. \quad (8)$$

By the definition of $F$ one can find $n_0 \in \mathbb{N}$ such that $m(A_{n_0}) > 0$ and

$$\|x_{n_0}\|_X > 1 - \frac{1}{k}. \quad (9)$$

Put $\Delta_n = [1 - 2^{-n}, 1 - 2^{-(n+1)}]$ for $n \leq k$. By composing suitable bijective transformations one can construct measurable mappings $g_n : [0, 1] \to [0, 1]$ and $\hat{g}_n : [0, 1] \to [0, 1]$ such that

$$g_n(A_{n_0}) \overset{m}{\sim} [0, 1] \setminus \Delta_n \quad \text{and} \quad g_n([0, 1] \setminus A_{n_0}) \overset{m}{\sim} \Delta_n, \quad (10)$$

the measure $\mu_n(\cdot) = m(g_n(\cdot)) : \Sigma \to \mathbb{R}$ is equivalent to $m$ \quad (11)

and

$$\hat{g}_n \circ g_n(t) = t \quad \text{for a.e. } t \in [0, 1] \quad (12)$$

for each $n \leq k$.

Next we will apply some observations which appear e.g. in [13] and [12]. Denote by $\Sigma \setminus m$ the quotient $\sigma$-algebra of Lebesgue measurable sets on $[0, 1]$ formed by identifying the $m$-null sets with $\emptyset$. Note that (11) gives in particular that the map $\phi_n : \Sigma \setminus m \to \Sigma \setminus m$ determined by $\phi_n(A) \overset{m}{\sim} g_n(A)$ for $A \in \Sigma$ is a Boolean isomorphism for each $n \leq k$. Observe that $\hat{g}_n(A) \overset{m}{\sim} \phi_n^{-1}(A)$ for $A \in \Sigma$ and $n \leq k$.

By (9) there are rotations $\{T_i\}_{i \leq N_k} \subset \mathcal{G}_X$ such that

$$\left\| x - \frac{1}{N_k} \sum_{i=1}^{N_k} T_i(x_{n_0}) \right\|_X < \frac{2}{k}. \quad (13)$$

According to (12) we may define mappings $S_i : L^\infty(X) \to L^\infty(X)$ for $n \leq k$ and $i \leq N_k$ by putting

$$S_i^{(n)}(F)(t) = T_i(F(\hat{g}_n(t))) \quad \text{for a.e. } t \in [0, 1], \quad F \in L^\infty(X).$$
By (11) we get that $S_i^{(n)}(t) \in \mathcal{G}_{L^\infty(X)}$ (see also [12, p. 467-468]).

The function $\chi_{[0,1]}x$ can be approximated by convex combinations as follows:

$$\left\| \chi_{[0,1]}x - \frac{1}{N} \sum_{i=1}^{k} \frac{1}{N_i} \sum_{i=1}^{N_k} S_i^{(n)}(F) \right\|_{L^\infty(X)} \leq \frac{1}{k} (2 + \sum_{i=1}^{k-1} 2k^{-1}).$$  \hfill (14)

Indeed, for $n \leq k$ and a.e. $t \in [0,1] \setminus \Delta_n$ it holds by (13) that

$$\left\| x - \frac{1}{N} \sum_{i=1}^{k} S_i^{(n)}(F)(t) \right\|_X = \left\| x - \frac{1}{N} \sum_{i=1}^{k} T_i^{(n)}(x_n) \right\|_X \leq \frac{2}{k}.$$  \hfill (15)

On the other hand, $\|x - \frac{1}{N} \sum_{i=1}^{k} S_i^{(n)}(F(t))\| \leq 2$ for a.e. $t \in \Delta_n$. In (14) we apply the fact that $\Delta_n$ are pairwise essentially disjoint.

Denote $A_2 = \frac{1}{k} \sum_{i=1}^{k} \frac{1}{N} \sum_{i=1}^{N_k} S_i^{(n)} \in \text{conv}(\mathcal{G}_{L^\infty(X)})$. By combining the estimates (8) and (14) we obtain by Fact 3.6 that

$$\|G - A_1 A_2(F)\| < \frac{5}{k}.$$  \hfill (16)

Observe that $A_1 A_2$ is an average of $N_k N_k$ many rotations on $L^\infty(X)$. We conclude by Fact 3.5 that $L^\infty(X)$ is uniformly convex-transitive.

The case $1 \leq p < \infty$ is a straightforward modification of the proof of [14, Thm. 2.1], where one replaces $U_i x_i$ by suitable averages belonging to $\text{conv}(\mathcal{G}_X(x_i))$ for each $i$. \hfill \Box

In fact it is not difficult to check the following fact: If the rotations of $L^\infty(X)$ are of the Banach-Stone type, then $L^\infty(X)$ is convex-transitive if and only if each $x \in S_X$ is a uniformly big point.

We already mentioned that $\ell^\infty/c_0$ is uniformly convex-transitive as a real space. Next we generalize this result to the vector-valued setting.

**Theorem 3.7.** Let $X$ be a uniformly convex-transitive Banach space over $\mathbb{K}$. Then $\ell^\infty(X)/c_0(X)$ (over $\mathbb{K}$) is uniformly convex-transitive.

**Proof.** Observe that the formula

$$T((x_n)_n) = (S_n x_{\pi(n)})_n,$$  \hfill (17)

where $\pi : \mathbb{N} \to \mathbb{N}$ is a bijection and $S_n \in \mathcal{G}_X$, $n \in \mathbb{N}$, defines a rotation on $\ell^\infty(X)$. Also note that such an isometry $T$ restricted to $c_0(X)$ is a member of $\mathcal{G}_{c_0(X)}$.

If $T \in \mathcal{G}_{\ell^\infty(X)}$ is as in (17), then $\widehat{T} : x + c_0(X) \mapsto T(x) + c_0(X)$, for $x \in \ell^\infty(X)$, defines a rotation $\ell^\infty(X)/c_0(X) \to \ell^\infty(X)/c_0(X)$. Indeed, it is clear that $\widehat{T} : \ell^\infty(X)/c_0(X) \to \ell^\infty(X)/c_0(X)$ is a linear bijection. Moreover,

$$\inf_{z \in c_0(X)} \|x - z\| = \inf_{z \in c_0(X)} \|T(x) - T(z)\| = \inf_{z \in c_0(X)} \|T(x) - z\|,$$

so that $\widehat{T} : \ell^\infty(X)/c_0(X) \to \ell^\infty(X)/c_0(X)$ is an isometry.
Fix $u, v \in S_{\ell^\infty(X)/c_0(X)}$. If $x, y \in \ell^\infty(X)$ are such that $u = x + c_0(X)$ and $v = y + c_0(X)$, then
\[
\text{dist}(x, c_0(X)) = \limsup_{n \to \infty} ||x_n|| = 1 = \text{dist}(y, c_0(X)) = \limsup_{n \to \infty} ||y_n||, \tag{16}
\]
since $u, v \in S_{\ell^\infty(X)/c_0(X)}$. Hence we may pick $x, y \in S_{\ell^\infty(X)}$ such that $u = x + c_0(X)$ and $v = y + c_0(X)$.

Fix $k \in \mathbb{N}$, $e \in S_{X}$ and let $A = \{n \in \mathbb{N} : ||x_n|| \geq 1 - \frac{1}{2k}\}$. Observe that $A$ is an infinite set by (16). Since $X$ is uniformly convex-transitive, there exists $N(k) \in \mathbb{N}$ such that for each $n \in A$ there are $T_1^{(n)}, \ldots, T_{N(k)}^{(n)} \in G_X$ such that
\[
\left| e - \frac{1}{N(k)} \sum_{l=1}^{N(k)} T_l^{(n)} x_n \right| < \frac{1}{k}. \tag{17}
\]

Fix $j(k) \in \mathbb{N}$ such that
\[
\frac{1}{j(k)} (2 + (j(k) - 1)(\frac{1}{k})) < \frac{2}{k}. \tag{18}
\]
Denote by $p_1, \ldots, p_{j(k)} \in \mathbb{N}$ the $j(k)$ first primes. Let $\phi_1, \ldots, \phi_{j(k)} : \mathbb{N} \to \mathbb{N}$ be permutations such that
\[
\phi_i(\mathbb{N} \setminus A) \subset \{p_i^m \mid m \in \mathbb{N}\} \quad \text{for} \ i \in \{1, \ldots, j(k)\}. \tag{19}
\]

For $l \in \{1, \ldots, N(k)\}$ put $S_{i,n,l} = T_l^{(\phi_i^{-1}(n))}$ if $\phi_i^{-1}(n) \in A$ and otherwise put $S_{i,n,l} = I$. Define a convex combination of rotations on $\ell^\infty(X)$ by letting
\[
A_1(z)|_n = \frac{1}{j(k)} \sum_{l=1}^{j(k)} \frac{1}{N(k)} \sum_{l=1}^{N(k)} S_{i,n,l}(z_{\phi_i^{-1}(n)}),
\]
where $(z_n)_{n \in \mathbb{N}} \in \ell^\infty(X)$. Consider $A_1 \in L(\ell^\infty(X))$ and $\overline{e} = (e, e, e, \ldots) \in \ell^\infty(X)$. We obtain that
\[
||\overline{e} - A_1((x_n))||_{\ell^\infty(X)} < \frac{2}{k}. \tag{20}
\]
Indeed, for each $n \in \mathbb{N}$ it holds for at least $j(k) - 1$ many indices $i$ that
\[
\frac{1}{N(k)} \sum_{l=1}^{N(k)} S_{i,n,l}(x_{\phi_i^{-1}(n)}) = \frac{1}{N(k)} \sum_{l=1}^{N(k)} T_l^{(\phi_i^{-1}(n))}(x_{\phi_i^{-1}(n)}),
\]
where one uses the definition of $S_{i,n,l}$ (19) and the fact that the sets $\{p_i^m \mid m \in \mathbb{N}\}, \{p_j^m \mid m \in \mathbb{N}\}$ are mutually disjoint for $i \neq j$. Thus (17) and (18) yield that
\[
\left| e - \frac{1}{j(k)} \sum_{l=1}^{j(k)} \frac{1}{N(k)} \sum_{l=1}^{N(k)} S_{i,n,l}(x_{\phi_i^{-1}(n)}) \right| < \frac{2}{k}
\]
holds for all $n \in \mathbb{N}$.

Next we will define another convex combination $A_2$ of rotations on $\ell^\infty(X)$ as follows. By using again the uniform convex transitivity of $X$ we obtain $T_{n,l} \in G_X, 1 \leq l \leq N(k), \ n \in \mathbb{N}$, such that
\[
\left| y_n - \frac{1}{N(k)} \sum_{l=1}^{N(k)} T_{n,l} e \right| < \frac{1}{k},
\]
holds for $n \in \mathbb{N}$. Define

$$A_2(z)_n = \frac{1}{N(k)} \sum_{l=1}^{N(k)} T_{n,l} z_n.$$ 

Combining the convex combinations yields

$$\|y - A_2 A_1 x\|_{\ell^\infty(X)} < \frac{3}{k}$$

according to Fact 3.6. Since the applied rotations induce rotations on $\ell^\infty(X)/c_0(X)$, we may consider the corresponding convex combinations in $L(\ell^\infty(X)/c_0(X))$ and thus

$$\|v - \hat{A}_2 \hat{A}_1 u\|_{\ell^\infty(X)/c_0(X)} < \frac{3}{k}.$$ 

Tracking the formation of the convex combinations reveals that $\hat{A}_2 \hat{A}_1$ can be written as an average of $N(k)j(k)N(k)$ many rotations on $\ell^\infty(X)/c_0(X)$. \qed

Since $C(\beta \mathbb{N} \setminus \mathbb{N})$ is linearly isometric to $\ell^\infty/c_0$, an application of Theorem 3.2 yields that $C(\beta \mathbb{N} \setminus \mathbb{N}, X)$ is uniformly convex-transitive if $X$ is uniformly convex-transitive. However, let us recall that this space is linearly isometric to $\ell^\infty(X)/c_0(X)$ if and only if $X$ is finite-dimensional.

4 Roughness and projections

Let $X$ be a Banach space. For each $x \in S_X$ we denote

$$\eta(X, x) = \limsup_{\|h\| \to 0} \frac{\|x + h\| + \|x - h\| - 2\|h\|}{\|h\|}.$$ 

Given $\varepsilon > 0$, the space $X$ is said to be $\varepsilon$-rough if $\inf_{x \in S_X} \eta(X, x) \geq \varepsilon$. In addition, 2-rough spaces are usually called extremely rough.

We will denote the coprojection constant of $X$ by

$$\rho(X) = \sup_{P} \|I - P\|,$$

where the supremum is taken over all linear norm-1 projections $P : X \to Y$.

A Banach space $X$ is called uniformly non-square if there exists $a \in (0, 1)$ such that if $x, y \in B_X$ and $\|x - y\| \geq 2a$ then $\|x + y\| < 2a$. These spaces were introduced in [15] by R. C. James, who also proved that this property lies strictly between uniform convexity and reflexivity. Next we will illustrate how the previous concepts are related.

**Theorem 4.1.** Let $X$ be a Banach space. Then the following conditions are equivalent:

(1) $X$ contains $\ell^1(2)$ almost isometrically.

(2) $X$ is not uniformly non-square.
Moreover, if \( \sup_{x \in X} \eta(X, x) = 2 \), then \( \rho(X) = 2 \).

We will require some preparations before the proof. Recall that given \( x, y \in X \) the function \( t \mapsto \frac{|x+t(y+\theta x)|-|x|}{t} \) is monotone in \( t \) and thus the limit \( \lim_{t \to 0^+} \frac{|x+t(y+\theta x)|-|x|}{t} \) exists and is finite.

**Lemma 4.2.** Let \( X \) be a Banach space and \( x, y \in X, \; x \neq 0 \). Then

\[
\lim_{t \to 0^+} \frac{|x+t(y+\theta x)|-|x|}{t} = \lim_{t \to 0^+} \frac{|x-t(y+\theta x)|-|x|}{t}
\]

for \( \theta = \lim_{t \to 0^+} \frac{|x-ty|-|x+ty|}{2t|y|} \).

**Proof.** Observe that for all maps \( a: [0, 1] \to \mathbb{R} \) such that \( \lim_{t \to 0^+} a(t) > 0 \) it holds that

\[
\lim_{t \to 0^+} \frac{|a(t)x+ty|-|a(t)x|}{t} = \lim_{t \to 0^+} \frac{|a(t)x+\frac{a(t)}{t}ty|-|a(t)x|}{t}
\]

\[
= \lim_{t \to 0^+} \frac{|x+\frac{1}{t}ty|-|x|}{t} = \lim_{t \to 0^+} \frac{|x+ty|-|x|}{t}.
\]

We will also apply the fact that

\[
\lim_{t \to 0^+} \frac{t(\lim_{t \to 0^+} \frac{|x-ty|-|x+ty|}{2t|y|}) - t \frac{|x-ty|-|x+ty|}{2t|y|}}{t} = 0.
\]

The claimed one-sided limits are calculated as follows:

\[
\lim_{t \to 0^+} \frac{|x + t(y + \theta x)| - |x|}{t}
\]

\[
= \lim_{t \to 0^+} \frac{|(1 + \frac{|x-ty|-|x+ty|}{2|y|})x + ty| - |x|}{t}
\]

\[
= \lim_{t \to 0^+} \frac{|(1 + \frac{|x-ty|-|x+ty|}{2|y|})x + ty| - (1 + \frac{|x-ty|-|x+ty|}{2|y|})||x||}{t}
\]

\[
+ \lim_{t \to 0^+} \frac{(1 + \frac{|x-ty|-|x+ty|}{2|y|})||x|| - |x|}{t}
\]

\[
= \lim_{t \to 0^+} \frac{|x + ty| - |x||}{t} + \lim_{t \to 0^+} \frac{|x - ty| - |x + ty|}{2t}
\]

\[
= \lim_{t \to 0^+} \frac{|x + ty| + |x - ty| - 2|x|}{2t}.
\]

In the first equality above we applied the fact (22), and in the third equality the fact (21). The calculations for the equation

\[
\lim_{t \to 0^+} \frac{|x - t(y + \theta x)| - |x|}{t} = \lim_{t \to 0^+} \frac{|x + ty| - |x - ty| - 2|x|}{2t}
\]

are similar. \( \square \)
Proof of Theorem 4.1. The equivalence of conditions (1) and (2) is well-known (see e.g. [9] or Remark 6.1 in [4]). The direction (1) \(\implies\) (3) is established by using the Hahn-Banach Theorem to obtain suitable rank-1 projections \(P\). Towards the implication (3) \(\implies\) (2), suppose that \(\rho(X) = 2\). Given \(\delta > 0\) there exists a projection \(P: X \to Y\), which satisfies \(||P|| = 1\) and \(||I - P|| > 2 - \frac{\delta}{2}\). Choose \(x \in S_X\) such that \(||x - P(x)|| > 2 - \frac{\delta}{2}\). This gives that \(||P(x)|| \geq 1 - \frac{\delta}{2}\). Put \(y = \frac{P(x)}{||P(x)||}\) and note that \(y \in S_X\) and \(||y - P(x)|| < \frac{\delta}{2}\). Moreover,

\[
||x - y|| \geq ||x - P(x)|| - ||y - P(x)|| > 2 - \delta > 2(1 - \delta)
\]
and

\[
||x + y|| \geq ||x + P(x)|| - ||y - P(x)|| > ||x + P(x)|| - \frac{\delta}{2}
\]

\[
= ||2x + P(x) - x|| ||P|| - \frac{\delta}{2}
\]

\[
\geq ||P(2x + P(x) - x)|| - \frac{\delta}{2} = ||P(2x)|| - \frac{\delta}{2} > 2 - \delta - \frac{\delta}{2} > 2(1 - \delta).
\]

Thus \(X\) is not uniformly non-square.

To verify the last sentence in the theorem, an application of Lemma 4.2 yields that if \(\sup_{x \in S_X} \eta(x, X) = 2\), then \(X\) is not uniformly non-square. Alternatively, this can be seen by modifying the argument in Remark 1 of [3]. We obtain that \(\rho(X) = 2\).

The extreme roughness of \(X\) is a tremendously stronger condition than \(\rho(X) = 2\). For example, if \((F_n)\) is a sequence of finite-dimensional smooth spaces such that \(\rho(F_n) \to 2\) as \(n \to \infty\), then the space

\[
X = \bigoplus_{n \in \mathbb{N}} F_n \quad \text{(summation in \(\ell^2\)-sense)}
\]

is Fréchet-smooth but \(\rho(X) = 2\).

However, for convex-transitive spaces \(X\) the condition of being extremely rough is equivalent to the condition \(\rho(X) = 2\). Indeed, if a convex-transitive space is not extremely rough then, by [5, Thm. 6.8], it must be uniformly convex and thus \(\rho(X) < 2\). It is unknown to us whether a convex-transitive Banach space is reflexive if it does not contain an isomorphic copy of \(\ell^1\).

In the same spirit as in this section, the projection constants of \(L^p\) spaces were discussed in [21].

5 Final Remarks: On the universality of transitivity properties

The well-known Banach-Mazur problem mentioned in the introduction asks whether every transitive, separable Banach space must be linearly isometric to a Hilbert space. It is well-known that all such (transitive+separable)
spaces must be smooth; otherwise, not much is known. Even adding some properties like being a dual space or even reflexivity has not sufficed, to date, for proving that the norm is Hilbertian.

Let us make a few remarks on the universality of some spaces of continuous functions. It is well-known that $C(\Delta)$ contains $C([0, 1])$ isometrically; hence, the former space is universal for the property of being uniformly convex-transitive and separable. However, it is not almost transitive.

To get a space which is universal for the property of being almost transitive and separable, just consider the almost transitive space $X = C^0_c(L)$ where $L$ is the pseudo-arc with one point removed ([16] or [19]). Since $[0, 1]$ is a continuous image of $L$, every separable space is isometrically contained in $X$ (complex case) or $X_\mathbb{R}$ (real case). Finally, note that the almost transitivity of a Banach space implies that of the real underlying space.

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ISSN 0784-3143 (print)
ISSN 1797-5867 (PDF)