BILIPSCHITZ EXTENDABILITY IN THE PLANE

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Abstract: We give a geometric characterization for a plane set $A \subset \mathbb{R}^2$ to have the following linear bilipschitz extension property: For $0 \leq \varepsilon \leq \delta$, every $(1 + \varepsilon)$-bilipschitz map $f: A \to \mathbb{R}^2$ has a $(1 + C\varepsilon)$-bilipschitz extension to the whole plane $\mathbb{R}^2$.

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1 Introduction

1.1. Let $A$ be a subset of the euclidean $n$-space $\mathbb{R}^n$ and let $L \geq 1$. A map $f: A \to \mathbb{R}^n$ is $L$-bilipschitz if

$$|x - y|/L \leq |f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in A$.

In general, an $L$-bilipschitz map $f: A \to \mathbb{R}^n$ cannot be extended to a bilipschitz map $F: \mathbb{R}^n \to \mathbb{R}^n$, not even to a homeomorphism, but this is often possible in the case the bilipschitz constant $L$ is close to 1.

1.2. Let $\Phi$ be the set of increasing homeomorphisms $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$. If $\varphi \in \Phi$ and $\delta > 0$, we say that a set $A \subset \mathbb{R}^n$ has the $(\varphi, \delta)$-bilipschitz extension property, $(\varphi, \delta)$-BLEP for short, if for $0 \leq \varepsilon \leq \delta$, every $(1 + \varepsilon)$-bilipschitz map $f: A \to \mathbb{R}^n$ has an extension to a $(1 + \varphi(\varepsilon))$-bilipschitz map $F: \mathbb{R}^n \to \mathbb{R}^n$. We say that a set $A \subset \mathbb{R}^n$ belongs to the class $\varphi$-BLEP if it has the $(\varphi, \delta)$-BLEP for some $\delta > 0$. In the case $\varphi(\varepsilon) = C\varepsilon$ we say that $A$ has the $(C, \delta)$-linear BLEP.

It was shown in [ATV2] that a set $A \subset \mathbb{R}^n$ has $(C, \delta)$-linear BLEP if it satisfies a geometric condition called sturdiness; see 2.2 for the definition. In this article we prove that the converse is true in the 2-dimensional case. More precisely, we obtain the following theorem.

1.3. Theorem. Let $A \subset \mathbb{R}^2$ contain at least three points. Then the following assertions are quantitatively equivalent:

- (1) $A$ is $c$-sturdy.
- (2) $A$ has the $(C, \delta)$-linear BLEP.

Here quantitative equivalence means that $C$ and $\delta$ depend only on $c$, and conversely, $c = c(C, \delta)$.

The proof is given in section 4.3. Note that a set $A \subset \mathbb{R}^n$ consisting of at most two points has the 1-linear BLEP but it is sturdy only in the cases $n = 1$ or $\#A = 1$.

For extension problems in higher dimensions and with more general bounds for the bilipschitz constant, see [Vä] and the references in [ATV2].

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2 Basic concepts

Notation follows closely our main reference [ATV2] and will not be repeated here except for the abbreviation $A(a, r) = A \cap B(a, r)$.

However, we recall three geometric properties of sets that are needed in our main result.
2.1. **Thickness.** For each unit vector \(e \in S^{n-1}\) we define the projection \(\pi_e : \mathbb{R}^n \rightarrow \mathbb{R}\) by \(\pi_e x = x \cdot e\). Let \(A \neq \emptyset\) be a bounded set in \(\mathbb{R}^n\). The *thickness* of \(A\) is the number

\[
\theta(A) = \inf \{d(\pi_e A) : e \in S^{n-1}\}.
\]

Alternatively, \(\theta(A)\) is the infimum of all \(t > 0\) such that \(A\) lies between two parallel hyperplanes \(F, F'\) with \(d(F, F') = t\). We have always \(0 \leq \theta(A) \leq d(A)\).

2.2. **Sturdiness.** Let \(A \subseteq \mathbb{R}^n\). For \(a \in A\) we set \(s(a) = s_A(a) = d(a, A \setminus \{a\})\). Then \(s(a) > 0\) if and only if \(a\) is isolated in \(A\).

Let \(c \geq 1\). We say that the set \(A \subseteq \mathbb{R}^n\) is \(c\)-sturdy if

1. \(\theta(A(a, r)) \geq 2r/c\) whenever \(a \in A\), \(r \geq cs(a)\), \(A \notin B(a, r)\),
2. \(\theta(A) \geq d(A)/c\).

If \(A\) is unbounded, we omit (2), and the condition \(A \notin B(a, r)\) of (1) is unnecessary.

2.3. **Relative connectivity** [TV, 4.6]. Let \(A \subseteq \mathbb{R}^n\) and \(M \geq 1\). A sequence \((x_0, x_1, \ldots, x_{N-1}, x_N)\) is proper if \(x_{j-1} \neq x_j\) for all \(j\). A sequence \((x_0, x_1, \ldots, x_{N-1}, x_N)\) in \(A\) is \(M\)-relative in \(A\) if it is proper and

\[
|x_{j-1} - x_j|/M \leq |x_j - x_{j+1}| \leq M|x_{j-1} - x_j|
\]

for all \(j\). Such a sequence is said to join the pairs \((x_0, x_1)\) and \((x_{N-1}, x_N)\). The set \(A\) is \(M\)-relatively connected (abbr. RC) if every two proper pairs in \(A\) can be joined by an \(M\)-relative sequence in \(A\).

The simplest examples of relatively connected sets are the connected ones, but also many totally disconnected sets like the Cantor middle-third set satisfy the RC-condition.

2.4. **Lemma.** Let \(A \subseteq \mathbb{R}^n\) be a closed \(c\)-sturdy set. Then \(A\) is \(c_1\)-RC for every \(c_1 > c\).

*Proof.* Let \(a \in A\) and \(r > 0\). Let \(c_1 > c\) and assume that \(A \cap \bar{B}(a, r) \neq \{a\}\) and \(A \notin \bar{B}(a, r)\). If \(R(a, r) = \{x \in A \mid r/c_1 \leq |x - a| \leq r\} = \emptyset\), then \(\theta(A(a, r)) \leq \theta(\bar{B}(a, r/c_1)) \leq 2r/c_1 < 2r/c\), a contradiction with the \(c\)-sturdiness of \(A\). It follows that, under the above assumptions, \(R(a, r) \neq \emptyset\), and by [TV, 4.11], this implies the claim. \(\square\)

2.5. **Linear isometric approximation property.** Let \(A \subseteq \mathbb{R}^n\). We say that \(A\) has the \((C, \delta)\)-linear isometric approximation property (IAP) if given \(0 < \varepsilon \leq \delta\), a \((1 + \varepsilon)\)-bilipschitz map \(f : A \rightarrow \mathbb{R}^n\), a point \(a \in A\) and \(r > 0\), there is an isometry \(T = T_{a,r} : \mathbb{R}^n \rightarrow \mathbb{R}^n\) such that

\[
|T x - f(x)| \leq C \varepsilon r
\]

for all \(x \in A \cap \bar{B}(a, r)\).
2.6. Theorem. Suppose that a set $A \subset \mathbb{R}^n$ has the $(C, \delta)$-linear BLEP. Then it has the $(C_1, \delta)$-linear IAP with $C_1 = C_1(C,n)$.

Proof. Let $f: A \to \mathbb{R}^n$ be $(1+\varepsilon)$-bilipschitz with $0 < \varepsilon \leq \delta$. Suppose that $a \in A$ and $r > 0$. Since $A$ has the $(C, \delta)$-linear BLEP, there is a $(1+C\varepsilon)$-bilipschitz extension $F: \mathbb{R}^n \to \mathbb{R}^n$ of $f$. Let $F_{a,r} = F \mid \bar{B}(a,r)$. Then $F_{a,r}$ is a $2C\varepsilon r$-nearisometry and since $\theta(\bar{B}(a,r)) = d(\bar{B}(a,r))$, [ATV1, 3.3] gives an isometry $T = T_{a,r}: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$||T - F_{a,r}||_{B(a,r)} \leq 2c_n C \varepsilon r.$$ 

In particular, we have $|Tx - f(x)| \leq 2C_n \varepsilon r$ for every $x \in A(a,r)$, and the proof is complete with $C_1 = 2c_n C$. □

3 Triangle maps

Since we work with the planar case, we use complex numbers whenever it simplifies notation.

3.1. The basic triangle map $f: \{-1,0,1\} \to \mathbb{R}^2$ is defined by

$$f(\pm 1) = \pm 1 \quad \text{and} \quad f(0) = i\sqrt{\varepsilon}.$$ 

This map is $(1 + \varepsilon)$-bilipschitz, but any approximation of $f$ by an isometry $T$ has an error at least $\sqrt{\varepsilon}/2$. This is seen by minimizing the distance from the image of $f$ to the straight line $TR$. The following elementary lemma generalizes this idea.

3.2. Lemma. Let $0 \leq \delta \leq \delta' \leq 1/4$, let $A = \{-1,a,1\} \subset \mathbb{R}^2$ be such that $\theta(A) = |a_2| \leq 2\delta$, and let $f: A \to \mathbb{R}^2$ satisfy $f(\pm 1) = \pm 1$ and $\theta(fA) = |f(a)_2| \geq 2\delta'$. If the disks $\bar{B}(\pm 1, \delta' - \delta)$ and $\bar{B}(f(a), \delta' + \delta)$ are disjoint, then every isometry $T: \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $||T - f||_A \geq \delta' - \delta$.

Proof. We emphasize that the conditions $\theta(A) = |a_2|$ and $\theta(fA) = |f(a)_2|$ belong to the assumptions. In particular, they imply that $-1 < a_1 < 1$ and $-1 < f(a)_1 < 1$ so that the situation is not too far from the basic map above.

Suppose that $T$ is an isometry with $||T - f||_A < \delta' - \delta$ and let $L = TR$. Writing $a' = (a_1, 0)$, we have

$$|Ta' - Ta| = |a' - a| = |a_2| \leq 2\delta.$$

If $L$ does not meet the disk $B(f(a), \delta' + \delta)$, then

$$|Ta - f(a)| \geq |Ta' - f(a)| - |Ta' - Ta| \geq (\delta' + \delta) - 2\delta = \delta' - \delta,$$

a contradiction.
It follows that the line $L$ meets all three disks $B(\pm1, \delta', \delta)$ and $B(f(a), \delta' + \delta)$. By assumption, these disks are disjoint, and by elementary geometry we get

$$(\delta' - \delta) + (\delta' + \delta) > |f(a)| = \theta(fA) \geq 2\delta',$$

which leads to a contradiction. The result follows from this. $\Box$

Later on we will need maps that are defined on a narrow neighbourhood of a line but that still possess the essential features of the basic triangle map: they should be $(1 + \varepsilon)$-bilipschitz but their approximation by isometries should produce an error of the order $\varepsilon$. The following lemmas show how to construct these maps.

**3.3. Lemma.** Let $0 \leq \varepsilon \leq 1/10$ and let $a, b \in [0, 1]$ be such that $2\varepsilon \leq a \leq b/2$. Then there is a $C^2$ function $f : \mathbb{R} \to \mathbb{R}$ satisfying

(i) $f(x) = 0$ for $x \leq 0$ and $x \geq b$;

(ii) $f(a) = \varepsilon^{3/2}$;

(iii) $f$ is $2\sqrt{\varepsilon}$-Lipschitz;

(iv) the curvature $K$ of the graph $y = f(x)$ satisfies $K \leq 1/\sqrt{\varepsilon}$.

**Proof.** Let $0 < o < a$ and consider first the interval $[o, a]$. One should think that $o \approx 0$, but we need $o > 0$ for technical reasons. Let $r = \sqrt{\varepsilon}$. The graph $y = f(x)$ consists of two circular arcs and a line segment. The construction is based on the diagram below, where also the notation is indicated.

![Diagram](attachment:image.png)

Part of the graph $y = f(x)$ with $h = \varepsilon \sqrt{\varepsilon}$.

By elementary geometry the variables $l$ and $\alpha$ must satisfy

$$\begin{cases} 2r \sin \alpha + l \cos \alpha = a - o \\ 2r(1 - \cos \alpha) + l \sin \alpha = \varepsilon^{3/2}, \end{cases}$$

and this system has the exact solution

$$l = \sqrt{(a - o)^2 - 4\varepsilon^2 + \varepsilon^3}, \quad \alpha = \arcsin(\sqrt{\varepsilon} (2(a - o) + l \varepsilon - 2l) / (l^2 + 4\varepsilon)).$$

The Lipschitz condition requires that $\tan \alpha \leq 2\sqrt{\varepsilon}$. It is geometrically obvious that $\alpha$ is decreasing in $a$, and thus $\alpha$ attains its maximum at $a = 2\varepsilon$. 

By substituting this value and choosing \( o \) small enough, we obtain \( \alpha \leq \arcsin \sqrt{\varepsilon} \leq \arctan(2\sqrt{\varepsilon}) \).

A similar construction is used on the interval \([a, b]\), and outside \([o, b - o]\) we define \( f(x) = 0 \). This function satisfies conditions (i)-(iv), but it is only piecewise \( C^2 \). However, at the six points where a circular arc is joined either to another arc or to a line segment, we use standard smoothing by clothoids (aka Cornu spirals), in an arbitrarily small neighbourhood of each joint, in such a way that the Lipschitz constant does not change, the curvature stays between the appropriate bounds, and the support of \( f \) does not expand outside \([0, b]\); see [Ad, p. 636] for the basic construction.

Using the following lemma we can construct tubular neighbourhood extensions for mappings of the type \( x \mapsto (x, f(x)) \).

3.4. Lemma. Let \( 0 < \varepsilon < 1/10 \), let \( I \subset \mathbb{R} \) be an interval and let \( f : I \to \mathbb{R} \) be \( \sqrt{\varepsilon} \)-Lipschitz and \( C^2 \). Define \( F : I \times [-\delta, \delta] \to \mathbb{R}^2 \) by setting

\[
F(x, y) = x + if(x) + yn(x),
\]

where \( n(x) \) is the upper unit normal to the graph \( y = f(x) \). Let \( K \) be the maximal curvature of \( y = f(x) \). If \( K\delta \leq \varepsilon \), then \( F \) is \((1 + 4\varepsilon)\)-bilipschitz. Moreover, if \( f(x) = 0 \) except for a subinterval of length \( l \), then \( |F(z) - z| \leq \sqrt{\varepsilon}l + \delta \) for every \( z \in I \times [-\delta, \delta] \).

Proof. Let \( z_i = (x_i, y_i) \in I \times [-\delta, \delta] \), \( i = 1, 2 \). Note that

\[
|y| \leq \delta, \quad |f'(x)| \leq \sqrt{\varepsilon} \quad \text{and} \quad \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} \leq K
\]

for all \((x, y)\).

In complex form we have

\[
n(x) = \frac{1}{\sqrt{1 + f'(x)^2}}(-f'(x) + i).
\]

Thus

\[
|F(z_1) - F(z_2)|^2 = |x_1 - x_2|^2 + \frac{y_1 f'(x_1)}{\sqrt{1 + f'(x_1)^2}} - \frac{y_2 f'(x_2)}{\sqrt{1 + f'(x_2)^2}})^2
\]

\[
-2(f(x_1) - f(x_2)) \left( \frac{y_1}{\sqrt{1 + f'(x_1)^2}} - \frac{y_2}{\sqrt{1 + f'(x_2)^2}} \right)
\]

\[
+2(f(x_1) - f(x_2)) \left( \frac{y_1}{\sqrt{1 + f'(x_1)^2}} - \frac{y_2}{\sqrt{1 + f'(x_2)^2}} \right) .
\]
Writing the right hand side above as \(|x_1 - x_2|^2 + t_1 + t_2 + t_3 + t_4\), where \(t_4\) contains the last two terms, we have to estimate each term. Since \(F\) is defined in a convex set, we can use the mean value theorem.

(i) To estimate \(t_1\), let \(g(x,y) = yf''(x)/\sqrt{1 + f'(x)^2}\). Then

\[
|\nabla g|^2 = \frac{y^2 f''(x)^2}{(1 + f'(x)^2)^3} + \frac{f'(x)^2}{1 + f'(x)^2} \leq \delta^2 K^2 + \varepsilon \leq 2\varepsilon,
\]

which implies that \(t_1 \leq 2\varepsilon |z_1 - z_2|^2\).

(ii) The upper bound \(t_2 \leq \varepsilon |x_1 - x_2|^2\) follows from the Lipschitz condition.

(iii) We need both upper and lower bounds for \(t_3\). Applying the mean value theorem for \(h(x,y) = y/\sqrt{1 + f'(x)^2}\), we get

\[
t_3 = \left( \frac{f'(u)}{(1 + f'(u)^2)^{3/2}}(x_1 - x_2) + \frac{1}{\sqrt{1 + f'(u)^2}}(y_1 - y_2) \right)^2
\]

where \((u,v)\) lies on the segment \([z_1, z_2]\). Using the estimate

\[
2\varepsilon^{3/2} |x_1 - x_2||y_1 - y_2| \leq 2\varepsilon |x_1 - x_2||y_1 - y_2| \leq \varepsilon |x_1 - x_2|^2 + \varepsilon |y_1 - y_2|^2,
\]

it follows that

\[
t_3 \leq \varepsilon K^2 \delta^2 |x_1 - x_2|^2 + \frac{1}{1 + f'(u)^2} |y_1 - y_2|^2 + 2\varepsilon K \delta |x_1 - x_2||y_1 - y_2|
\]

\[
\leq \varepsilon^3 |x_1 - x_2|^2 + |y_1 - y_2|^2 + \varepsilon |x_1 - x_2|^2 + \varepsilon |y_1 - y_2|^2
\]

\[
\leq 2\varepsilon |x_1 - x_2|^2 + (1 + \varepsilon)|y_1 - y_2|^2.
\]

In the opposite direction, we have

\[
t_3 \geq \frac{1}{1 + \varepsilon} |y_1 - y_2|^2 - 2\varepsilon K \delta |x_1 - x_2||y_1 - y_2|
\]

\[
\geq (1 - 2\varepsilon)|y_1 - y_2|^2 - \varepsilon |x_1 - x_2|^2.
\]

(iv) Rearranging and using the Taylor formula, we have

\[
t_4 = \frac{2y_1}{\sqrt{1 + f'(x_1)^2}}(f(x_1) - f(x_2) - f'(x_1)(x_1 - x_2))
\]

\[
+ \frac{2y_2}{\sqrt{1 + f'(x_2)^2}}(f'(x_2)(x_1 - x_2) - f(x_1) + f(x_2))
\]

\[
= \left( \frac{y_1 f''(\xi_1)}{\sqrt{1 + f'(x_1)^2}} - \frac{y_2 f''(\xi_2)}{\sqrt{1 + f'(x_2)^2}} \right) |x_1 - x_2|^2,
\]

where \(\xi_1, \xi_2 \in [x_1, x_2]\). Since \(|f''(\xi)| \leq (1 + \varepsilon)^{3/2}\), this implies that

\[
|t_4| \leq 2 K \delta (1 + \varepsilon)^{3/2} |x_1 - x_2|^2 \leq 3\varepsilon |x_1 - x_2|^2.
\]
Using these estimates we obtain

\[
|F(z_1) - F(z_2)|^2 \leq \left| x_1 - x_2 \right|^2 + 2\varepsilon |x_1 - x_2|^2 + 2\varepsilon |y_1 - y_2|^2 + \varepsilon |x_1 - x_2|^2 \\
+ 2\varepsilon |x_1 - x_2|^2 + (1 + \varepsilon) |y_1 - y_2|^2 + 3\varepsilon |x_1 - x_2|^2 \\
= (1 + 8\varepsilon )|x_1 - x_2|^2 + (1 + 3\varepsilon)|y_1 - y_2|^2.
\]

so that \(|F(z_1) - F(z_2)| \leq \sqrt{1 + 8\varepsilon} |z_1 - z_2| \leq (1 + 4\varepsilon) |z_1 - z_2|.

For the lower bound, we discard irrelevant positive terms and get

\[
|F(z_1) - F(z_2)|^2 \geq |x_1 - x_2|^2 + t_3 - |t_4| \\
\geq (1 - 4\varepsilon) |x_1 - x_2|^2 + (1 - 2\varepsilon) |y_1 - y_2|^2 \\
\geq (1 - 4\varepsilon) |z_1 - z_2|^2.
\]

This implies that \(|F(z_1) - F(z_2)| \geq \sqrt{1 - 4\varepsilon} |z_1 - z_2| \geq |z_1 - z_2|/(1 + 4\varepsilon).

The proof for the bilipschitz condition is now complete, and the last inequality is obvious. □

3.5. Lemma. Let \(A \subset \mathbb{R}^n\) and let \(\varepsilon \leq 1/10\). Suppose that \(a \in A, r > 0\) and let \(f : A \to \mathbb{R}^n\) be \((1 + \varepsilon)\)-bilipschitz such that \(|f(z) - z| \leq \varepsilon r\) whenever \(|z - a| \leq r/2\) and \(f(z) = z\) for \(|z - a| \geq r/2\). Define \(F : A \cup (\mathbb{R}^n \setminus B(a, r)) \to \mathbb{R}^n\) by setting

\[
F(z) = \begin{cases} 
  f(z) & \text{for } z \in A, \\
  z & \text{for } |z - a| \geq r.
\end{cases}
\]

Then \(F\) is \((1 + 3\varepsilon)\)-bilipschitz.

Proof. Let \(z_1 \in A \cap B(a, r/2)\) and \(|z_2 - a| \geq r\). Then \(|z_1 - z_2| \geq r/2\), which implies that

\[
|F(z_1) - F(z_2)| = |f(z_1) - z_2| \leq |f(z_1) - z_1| + |z_1 - z_2| \leq \varepsilon r + |z_1 - z_2| \\
\leq (1 + 2\varepsilon)|z_1 - z_2|.
\]

In the opposite direction, we have

\[
|F(z_1) - F(z_2)| = |f(z_1) - z_2| \geq |z_1 - z_2| - |f(z_1) - z_1| \geq |z_1 - z_2| - \varepsilon r \\
\geq (1 - 2\varepsilon)|z_1 - z_2| \geq |z_1 - z_2|/(1 + 3\varepsilon),
\]

since \(\varepsilon \leq 1/10\).

All other cases for \(z_1, z_2\) are trivial, and the proof is complete. □

Finally, we need an estimate on the distortion of angles under bilipschitz maps.

3.6. Lemma. Let \(1 < t \leq 2\) and let \(f : \{0, 1, t\} \to \mathbb{R}^n\) be \((1 + \varepsilon)\)-bilipschitz with \(\varepsilon \leq 1/100\). Let \(A = f(0), B = f(1), C = f(t)\) and \(\alpha = \angle BAC\). Then \(\alpha \leq 2.1\sqrt{\varepsilon}\).
Proof. Consider the triangle with vertices $A, B, C$. Elementary geometrical considerations show that $\alpha$ is maximal in the case $AB = 1 + \varepsilon$, $BC = (t - 1)(1 + \varepsilon)$, and $AC = t/(1 + \varepsilon)$. Using trigonometry and Taylor approximation we obtain

$$\sin \alpha \leq 2\sqrt{(t - 1)\varepsilon} \leq 2\sqrt{\varepsilon} \leq 0.2.$$ 

Furthermore, for these values we have $\alpha \leq 1.01 \sin \alpha \leq 2.1\sqrt{\varepsilon}$, and the proof is complete. $\square$

4 Main proofs

We use triangle maps to prove the following theorem, which constitutes the first part of our main result.

4.1 Theorem. Let $\lambda \geq 1$, $c > (30\lambda)^8$, and let $A \subset \mathbb{R}^2$ be $\lambda$-relatively connected but not $c$-sturdy. Then for $1/\sqrt{c} \leq \varepsilon \leq 1/(30\lambda)^4$ there is a $(1 + 48\varepsilon)$-BL map $f: A \rightarrow \mathbb{R}^2$ with the following property: there are $a \in A$ and $r > 0$ such that

$$||T - f||_{A(a, r)} \geq \left(\frac{r}{6000\lambda^3}\right)\sqrt{\varepsilon}$$

for all isometries $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

Proof. Since $A$ is not $(1/\varepsilon^2)$-sturdy, there are two possibilities.

Case 1: Condition 2.2(1) is not satisfied. In this case there are $a \in A$ and $r > 0$ such that $A \not\subset B(a, r)$, $s(a) \leq \varepsilon^2 r$ and $\theta(A(a, r)) \leq 2\varepsilon^2 r$. By scaling, we may assume that $a = 0$, $r = 1$, and then $A \not\subset B(1) = B(0, 1)$, $s(0) \leq \varepsilon^2$, $\theta(A(0, 1)) \leq 2\varepsilon^2$. Furthermore, we may assume that $A(0, 1)$ is contained in the $2\varepsilon^2$-neighbourhood of $\mathbb{R} \subset \mathbb{R}^2$.

We apply [TV, 4.11(2)] with $c = 4\lambda$ to find points $u, v \in A$ as follows. Since $s(0) \leq \varepsilon^2 < \varepsilon$, the set $A(0, 2\varepsilon)$ contains at least two points. Also $A \not\subset B(1)$, and thus there is a point $u \in A \cap B(8\varepsilon) \setminus B(2\varepsilon)$. Similarly, since $80\lambda^2 \varepsilon \leq 1$, there is $v \in A \cap B(80\lambda^2 \varepsilon) \setminus B(20\lambda \varepsilon)$. There are six possibilities for the order of the points $0, u_1, v_1$ and of these only two are essentially different; we consider the case where $0 < u_1 < v_1 < 1$, the other cases being similar. However, the constants appearing below apply for all cases and may thus seem unnecessarily large for this special case.

We construct a bilipschitz map $f: A \rightarrow \mathbb{R}^2$ as follows:

- Apply Lemma 3.3 with substitutions $0 \mapsto 0$, $a \mapsto u_1$, $b \mapsto v_1$. This gives a $2\sqrt{\varepsilon}$-Lipschitz map $f_1: \mathbb{R} \rightarrow \mathbb{R}$ such that $f_1(x) = 0$ if $x \not\in [0, v_1]$, $f_1(u_1) = \varepsilon^{3/2}$, and $K \leq 1/\sqrt{\varepsilon}$.

- Apply Lemma 3.4 with $\varepsilon \mapsto 4\varepsilon$, $\delta \mapsto 2\varepsilon^2$, $I \mapsto \mathbb{R}$ and $f \mapsto f_1$. Then $K\delta \leq 2\varepsilon^{3/2} \leq 4\varepsilon$, and the resulting map $F: \mathbb{R} \times [-\delta, \delta] \rightarrow \mathbb{R}^2$ is $(1 + 16\varepsilon)$-BL. Also, we have $l \leq 160\lambda^2\varepsilon$ and therefore

$$|F(z) - z| \leq 160\lambda^2\varepsilon^{3/2} + 2\varepsilon^2 < \varepsilon$$

for all $z$. 

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• We extend the definition of $F$ outside $B(1)$ by $F(z) = z$. Substitute $\varepsilon \mapsto 16\varepsilon$ and $r = 1/2$ in Lemma 3.5. Since $160\lambda^2 \varepsilon \leq r/2$, we have $|F(z) - z| \leq \varepsilon \leq 16\varepsilon r$ for $|z| \leq r/2$ and $F(z) = z$ for $|z| \geq r/2$. It follows that $F$ is $(1 + 48\varepsilon)$-BL.

• The domain of definition for $F$ contains the set $A$ and by restriction we get the required $(1 + 48\varepsilon)$-BL map $f: A \to \mathbb{R}^2$.

It remains to show that $f$ cannot be well approximated by isometries. For this it suffices to consider the restriction $f \mid \{0, u, v\}$ in the disk $B = B(0, r_1)$, where $r_1 = 160\lambda^2 \varepsilon$. Let $A' = \{0, u, v\}$ and let $h: \mathbb{R}^2 \to \mathbb{R}^2$ be a similarity such that $h(0) = -1, h(v) = 1$ and let $g = hfh^{-1}: A' \to hA'$. Since

$$f(0) = 0, f(v) = v, \text{Lemma 3.2 can be applied to } g. \text{ The similarity ratio } t \text{ of } h \text{ satisfies } 1/80\lambda^2 \varepsilon \leq t \leq 1/10\lambda \varepsilon, \text{ and thus } \theta(hA') \leq 2\varepsilon^2/10\lambda \varepsilon = \varepsilon/5\lambda \text{ and } \theta(ghA')) \geq (\varepsilon^{3/2} - 2\varepsilon^2)/160\lambda^2 \varepsilon > \sqrt{8}/162\lambda^2. \text{ Thus the error of approximation of } g \text{ by an isometry is at least }$$

$$\sqrt{8}/324\lambda^2 - \varepsilon/10\lambda \geq \sqrt{8}/340\lambda^2,$$

and therefore

$$||T - f||_{A(0, r_1)} \geq 10\lambda \varepsilon (\sqrt{8}/340\lambda^2) = \varepsilon^{3/2}/34\lambda = \frac{r_1}{6000\lambda^3 \sqrt{8}},$$

for all isometries $T$. This completes the proof for Case 1.

Case 2: Condition 2.2(2) is not satisfied. This implies that $A$ is bounded and $\theta(A) < \varepsilon^2 d(A)$. Using $\lambda$-relative connectedness, we can find points $a, b, c \in A$ such that $1 \leq |a - b|/|b - c| \leq \lambda$. Using Lemmas 3.3 and 3.4, we can construct a map $f: A \to \mathbb{R}^2$ that by 3.2 contradicts the requirements. The details are similar to Case 1 and are omitted.

This completes the proof. □

4.2. Theorem. Let $\lambda \geq 1000, \text{ let } A \subset \mathbb{R}^n \text{ be a closed set that is not } \lambda \text{-relatively connected. Then there is } \varepsilon \leq 2/(\lambda - 2) \text{ and a } (1 + \varepsilon) \text{-bilipschitz map } f: A \to \mathbb{R}^n \text{ with the following property: If } F: \mathbb{R}^n \to \mathbb{R}^n \text{ is a } (1 + \delta) \text{-bilipschitz extension of } f, \text{ then } \delta \geq 1/20 \ln^2 \varepsilon.$$

Proof. We use the concept of upper sets from [TV, 4.9]. Since $A$ is not $\lambda$-relatively connected, the upper set $\hat{A}$ consists of more than one $\ln \lambda$-component. Let $\gamma$ be a $\ln \lambda$-component that is not the greatest element; see [TV, 3.2]. By [TV, 3.4(11) and 3.4(14)] the set $\pi \gamma$ is compact, and by [TV, 3.4(12)] we have $A \cap B(\pi \gamma, (\lambda - 1)d(\pi \gamma)) = \pi \gamma$. Choose $a, b, c \in \pi \gamma$ such that $|a - b| = d(\pi \gamma)$ and then $z \in A \setminus \pi \gamma$ such that $d(z, \pi \gamma)$ is minimal. We may assume that $|b - z| \leq |a - z|$, and hence $\angle abz \geq \pi/3$. Using suitable similarities, we may assume that $b = 0, |a - b| = 1$ and $z = te_1$ with $t \geq \lambda - 1$.

We choose $\varepsilon = 2/(t - 1) \leq 2/(\lambda - 2) < 0.01$ and construct a $(1 + \varepsilon)$-bilipschitz map $f: A \to \mathbb{R}^n$ as follows. Let $f | (A \setminus B(0, 1)) = \text{id}$, and let $f$
rotate $B(0,1)$ so that $f(0) = 0$ and $f(a) = e_1$. To calculate the bilipschitz constant $L$ of $f$, we note that the worst case arises from $a = -e_1$, $f(a) = e_1$; this seems geometrically obvious and can be proved by solving an elementary extremal value problem. Thus

$$L \leq \frac{t+1}{t-1} = 1 + \frac{2}{t-1} = 1 + \varepsilon.$$

Suppose now that $f$ can be extended to a $(1+\delta)$-bilipschitz map $F: \mathbb{R}^n \to \mathbb{R}^n$. We apply Lemma 3.6 to the map $F^{-1} \mid \{0, e_1, 2e_1, 4e_1, \ldots, 2^N e_1, z\}$, where $N = [\log_2 t]$. Let $a_i = F^{-1}(2^i e_1)$ for $i = 0, 1, 2, \ldots, N$ and $a_{N+1} = z$. The lemma implies that

$$\sum_{i=0}^N \angle a_i a_{i+1} \leq 2 \sqrt{\delta} (N + 1) \leq 2 \sqrt{\delta} (\log_2 t + 1)$$

Since $t = \frac{2}{\varepsilon} + 1 \leq 2.1/\varepsilon$, we obtain

$$\delta \geq \frac{1}{10 \ln^2 (4.2/\varepsilon)} \geq \frac{1}{20 \ln^2 \varepsilon}.$$

This completes the proof. □

4.3. Proof of Theorem 1.3. The implication $(1) \Rightarrow (2)$ was the main result of [ATV2].

For the converse part, suppose that $A$ has the $(C, \delta)$-linear BLEP. Let $\lambda \geq 2/\delta + 2$ so that $\varepsilon = 2/(\lambda - 2) \leq \delta$ and suppose that $A$ is not $\lambda$-RC. We may assume that $\lambda \geq 1000$. Let $f: A \to \mathbb{R}^2$ be the $(1+\varepsilon)$-bilipschitz map given by Theorem 4.2. Since $A$ has the $(C, \delta)$-linear BLEP, we have

$$C \varepsilon \geq \frac{1}{\sqrt{\varepsilon}}$$

This leads to a contradiction unless $\varepsilon \geq \varepsilon(C) > 0$, which is equivalent to $\lambda \leq (C, \delta) < \infty$.

It follows that $A$ is $\lambda$-relatively connected with the above bound.

By Theorem 2.6 the set $A$ has the $(C_1, \delta)$-IAP with $C_1 = C_1(C)$. Supposing that $A$ is not $c$-sturdy, we must find an upper bound for $c$, and may thus assume that $c > (30\lambda)^8 \vee 48^2/\delta^2$. Let $\varepsilon = 1/\sqrt{c}$ so that $48\varepsilon \leq \delta$. Applying Theorem 4.1, we obtain a $(1+48\varepsilon)$-bilipschitz map $f: A \to \mathbb{R}^2$ such that

$$\|T - f\|_{A(a,r)} \geq (r/6000L^3)\sqrt{\varepsilon}$$

for some $a \in A$, $r > 0$ and for all isometries $T$. The IAP of $A$ thus leads to the estimate

$$C_1 \cdot 48\varepsilon \geq \sqrt{\varepsilon}/6000L^3,$$

which is a contradiction unless $\varepsilon \geq \varepsilon(C, \lambda)$, or equivalently, unless $c \leq c(C, \delta)$.

It follows that $A$ is $c$-sturdy with the above bound, and the proof of the main theorem is complete. □
4.4. Remark. The first part of the above proof can be easily modified to show that a planar set $A$ having the $\varphi$-BLEP is relatively connected if

$$\lim_{\varepsilon \to 0} \varphi(\varepsilon) \ln^2 \varepsilon = 0.$$ 

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