APPROXIMATION OF THE LAPLACE TRANSFORM
BY THE CAYLEY TRANSFORM

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Abstract: We interpret the usual Cayley transform of linear (infinite-dimensional) state space systems as a numerical integration scheme of Crank–Nicholson type. This turns out to be equivalent to an approximation procedure of the Laplace transform. The convergence properties of such an approximation are investigated.

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1 Introduction and motivation

Let $U$ and $X$ be separable Hilbert spaces. Let $S = [\begin{bmatrix} A & B \\ C & D \end{bmatrix}]$ be a system node in the sense of [8], whose input and output space are $U$, and the state space is $X$. An additional space $V := \{ [\begin{bmatrix} x \\ u \end{bmatrix}] \in [X \oplus U] : A_{-1}x + Bu \in X \}$ is defined as usual, and it is equipped with the natural norm making it a Hilbert space.

Then, as is well-known, the Cauchy problem

$$\begin{cases} x'(t) = A_{-1}x(t) + Bu(t), & t \geq 0, \\ x(0) = x_0 \end{cases}$$

is uniquely solvable for any input $u \in C^2(\mathbb{R}_+; U)$ and initial state $x_0 \in X$ for which the compatibility condition $[x_0 \ u(0)] \in V$ holds. Moreover, then also $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in C(\mathbb{R}_+; V)$, and hence the output relation $y(t) = C&D \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$ is well defined for all $t \geq 0$ as $\mathcal{C}D \in \mathcal{L}(V; U)$. These and many other facts can be found in [8, Section 2].

The system node $[\begin{bmatrix} A & B \\ C & D \end{bmatrix}]$ is energy preserving if the following energy balance holds for all $T > 0$

$$\langle x(T), x(T) \rangle_X + \int_0^T \langle y(t), y(t) \rangle_Y dt = \langle x_0, x_0 \rangle_X + \int_0^T \langle u(t), u(t) \rangle_Y dt, \quad (1.2)$$

where $u$, $x$, $y$ and $x_0$ are as in (1.1). For any energy preserving $S$, the semigroup generator $A$ is maximally dissipative and $\mathbb{C}_+ \subset \rho(A)$. If both $S = [\begin{bmatrix} A & B \\ C & D \end{bmatrix}]$ and its dual node $S^d = [\begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix}]$ are energy-preserving, then $[\begin{bmatrix} A & B \\ C & D \end{bmatrix}]$ is called conservative; see [8, Definitions 3.1 and 4.1]. Conservative system nodes are known in classical operator theory as operator colligations or Livšic–Brodskiǐ nodes. A wide classical literature exists for them but the practical linear systems content might sometimes be hard to understand. See e.g. Brodskiǐ [4, 6, 5], Livšic [12], Livšic and Yantsevich [11], Sz.-Nagy and Foiaş [15], Smuljan [13], and Helton [3]. An up-to-date, comprehensive reference for operator nodes is Staffans [14]. The general conservative case is treated in Malinen, Staffans and Weiss [8], and the special case of boundary control systems are described in [7, 9].

For simplicity, it will be henceforth assumed that all system nodes treated in this paper are conservative, even though most of the results could be given in a more general setting. For the same reason, we assume that $U = \mathbb{C}$, i.e. the signals $u(\cdot)$ and $y(\cdot)$ in (1.1) are scalar valued, even though everything would still remain true (with similar proofs) even if $U$ was a separable Hilbert space.

Let us assume, for a moment, that we are treating the matrix case. Then the dynamical equations take the usual form

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), & t \geq 0, \\ x(0) = x_0. \end{cases} \quad (1.3)$$
where $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times 1}$, $C \in \mathbb{C}^{1 \times n}$, and $D \in \mathbb{C}$. Let $h > 0$ be a discretization parameter. We can carry out a slightly nonstandard time discretization of (1.3) and obtain an approximation of Crank–Nicholson type

$$
\begin{align*}
  x(jh) - x((j-1)h) &\approx A \frac{x(jh) + x((j-1)h)}{2} + Bu(jh), \\
  y(jh) &\approx C \frac{x(jh) + x((j-1)h)}{2} + Du((j-1)h), \quad j \geq 1, \\
  x(0) &= x_0.
\end{align*}
$$

Clearly, this induces the discrete time dynamics

$$
\begin{align*}
  x_j^{(h)} - x_{j-1}^{(h)} &= A \frac{x_j^{(h)} + x_{j-1}^{(h)}}{2} + Bu_j^{(h)}, \\
  y_j^{(h)} &= C \frac{x_j^{(h)} + x_{j-1}^{(h)}}{2} + Du_j^{(h)}, \quad j \geq 1, \\
  x_0^{(h)} &= x_0,
\end{align*}
$$

where loosely speaking $u_j^{(h)}/\sqrt{h}$ is an approximation of $u(jh)$. We hope very much that $y_j^{(h)}/\sqrt{h}$ would be close to $y(jh)$ — at least under some exceptionally happy circumstances. After some easy computations, equations (1.4) take the form

$$
\begin{align*}
  x_j^{(h)} &= A_x x_{j-1}^{(h)} + B_x u_j^{(h)}, \\
  y_j^{(h)} &= C_x x_{j-1}^{(h)} + D_x u_j^{(h)}, \quad j \geq 1, \\
  x_0^{(h)} &= x_0,
\end{align*}
$$

where $A_x := (\sigma + A)(\sigma - A)^{-1}$, $B_x := \sqrt{2\sigma}(\sigma - A)^{-1}B$, $C_x := \sqrt{2\sigma}C(\sigma - A)^{-1}$ and $D_x := D + C(\sigma - A)^{-1}B$ with $\sigma := 2/h$.

Even though the computation leading to (1.5) was carried out in the matrix setting, exactly the same transformation can be done for any system node $S = [A B; C D]$. We simply define the discrete time linear system (henceforth, DLS) described by the operator quadruple

$$
\phi_\sigma = \begin{bmatrix} A_\sigma & B_\sigma \\ C_\sigma & D_\sigma \end{bmatrix} = \begin{bmatrix} (\sigma + A)(\sigma - A)^{-1} & \sqrt{2\sigma}(\sigma - A)^{-1}B \\ \sqrt{2\sigma}C(\sigma - A)^{-1} & G(\sigma) \end{bmatrix} \quad (1.6)
$$

for any $\sigma > 0$ (or even for any $\sigma \in \mathbb{D}$, $\mathbb{D}$ being the unit disk, but we shall not use this in this paper). Here $G(\cdot)$ denotes the transfer function of $S$, and it is defined by $G(s) = C&D [ (s-I)^{-1}B I]^T$ for all $s \in \mathbb{C}_+$.

In system theory, the transformation $S \mapsto \phi_\sigma$ is called Cayley transform of continuous time systems to discrete time systems. By some computations, it can be checked that the discrete time transfer function $D_\sigma(\cdot)$ of $\phi_\sigma$ satisfies

$$
D_\sigma(z) := D_\sigma + zC_\sigma(I-zA_\sigma)^{-1}B_\sigma = G \left( \frac{1-z}{1+z} \right). \quad (1.7)
$$

We say that the DLS $\phi_\sigma$ of type (1.5) is conservative if the defining block matrix $[A_x B_x; C_x D_x]$ is unitary. Then the discrete time balance equation

$$
\sum_{j=1}^N \|x_j\|^2 - \sum_{j=1}^N \|x_{j-1}\|^2 = \sum_{j=1}^N \|u_j\|^2 - \sum_{j=1}^N \|y_{j-1}\|^2
$$
is satisfied for all $N \geq 1$, where the sequences $\{u_j\}$, $\{x_j\}$ and $\{y_j\}$ satisfy (1.5). Studying the approximation scheme (1.4) might not be well motivated, unless the following proposition did not hold:

**Proposition 1.** Let the system node $S = [\begin{bmatrix} A & B \\ C & D \end{bmatrix}]$ and the DLS $\phi_\sigma = [\begin{bmatrix} A_\sigma & B_\sigma \\ C_\sigma & D_\sigma \end{bmatrix}]$ be connected by (1.6). Then $S$ is (continuous time) conservative (passive) if and only if $\phi_\sigma$ is (discrete time) conservative (resp., passive).

There exists an extensive literature on the Cayley transform of systems, and we shall not try to make a full account of it here. See e.g. Ober and Montgomery-Smith [10]. A nice piece of work, parallelling our approach, is Arov and Gavrilyuk [1].

## 2 Approximation of the input/output mapping

In this section, we describe the discretization (1.5) of dynamical system (1.1) in operator theory language.

### 2.1 Spaces and transforms.

The norm of the usual Hardy space $H^2(\mathbb{C}_+)$ is given by

$$
\|\Phi\|_{H^2(\mathbb{C}_+)}^2 = \sup_{x > 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi(x + yi)|^2 \, dy.
$$

As usual, the Laplace transform is defined

$$(\mathcal{L}f)(s) = \int_0^{\infty} e^{-st} f(t) \, dt \quad \text{for all} \quad s \in \mathbb{C}_+,$$

and it maps $L^2(\mathbb{R}_+) \rightarrow H^2(\mathbb{C}_+)$ unitarily. The norm of $H^2(\mathbb{D})$ is given by

$$
\|\phi\|_{H^2(\mathbb{D})}^2 = \sum_{j \geq 0} |\phi_j|^2 \quad \text{if} \quad \phi(z) = \sum_{j \geq 0} \phi_j z^j,
$$

which makes the $Z$-transform unitary from $l^2(\mathbb{Z}_+) \rightarrow H^2(\mathbb{D})$. If, say, $f \in C_c(\mathbb{R})$ in (2.1), then $(\mathcal{L}f)(s)$ is well defined for all $s \in i\mathbb{R}$, too. We then call the function $i\omega \mapsto (\mathcal{L}f)(i\omega)$ the Fourier transform of $f$.

From now on, denote by $D_\sigma : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ the multiplication operator defined by $(D_\sigma \hat{u})(z) = D_\sigma(z)\hat{u}(z)$ for all $z \in \mathbb{D}$ and $\sigma > 0$. Similarly, denote by $G : H^2(\mathbb{C}_+) \rightarrow H^2(\mathbb{C}_+)$ the multiplication operator satisfying

$$(G\hat{u})(s) = G(s)\hat{u}(s) \quad \text{for all} \quad s \in \mathbb{C}_+^2.$$

It follows immediately that (1.7) takes the form of the similarity transformation

$$
G = C_\sigma^{-1} D_\sigma C_\sigma,
$$

where the *composition operator* is defined by $(C_\sigma F)(z) := F(\frac{1-z\bar{\sigma}}{1-z})$ for all $z \in \mathbb{D}$ and $F : \mathbb{C}_+ \rightarrow \mathbb{C}$. Trivially $(C_\sigma^{-1} f)(s) := f(\frac{s-z\bar{\sigma}}{s-\sigma})$ for all $s \in \mathbb{C}_+$ and $f : \mathbb{D} \rightarrow \mathbb{C}$. \(^2\)

---

\(^2\)Then $D_\sigma$ and $G$ are unitarily equivalent to the input/output mappings of $\phi_\sigma$ and $S$, respectively.
Proposition 2. The mapping \( f \mapsto F \) given by \( F(s) = \frac{\sqrt{2/\sigma}}{1+\sigma/\sigma} f(\frac{s-\sigma}{\sigma}) \) is unitary from \( H^2(\mathbb{D}) \) onto \( H^2(\mathbb{C}_+) \). In particular, the operator \( M_\sigma \mathcal{C}_\sigma^{-1} : H^2(\mathbb{D}) \to \) \( H^2(\mathbb{C}_+) \) is unitary, where \( M_\sigma : H(\mathbb{C}_+) \to H(\mathbb{C}_+) \) denotes the multiplication operator by \( \frac{\sqrt{2/\sigma}}{1+\sigma/\sigma} \).

Proof. This follows as soon as it is shown that for each \( \sigma > 0 \), the sequence 
\[ \left\{ \frac{\sqrt{2/\sigma}}{1+\sigma/\sigma} \left( \frac{s-\sigma}{\sigma} \right)^j \right\}_{j \geq 0} \]

is an orthonormal basis for \( H^2(\mathbb{C}_+) \). \( \Box \)

2.2 Discretizing operators.

By \( T_\sigma \) we denote a discretizing (or sampling) bounded linear operator \( T_\sigma : L^2(\mathbb{R}_+) \to H^2(\mathbb{D}) \). The adjoint \( T_\sigma^* \) of \( T_\sigma \) maps then \( H^2(\mathbb{D}) \to L^2(\mathbb{R}_+) \), and it is typically an interpolating operator. In this paper, we define \( T_\sigma \) as by

\[ (T_\sigma u)(z) = \sum_{j \geq 1} u_j^{(h)} z^j \quad \text{where} \quad u_j^{(h)} = \frac{1}{\sqrt{h}} \int_{(j-1)h}^{jh} u(t) \, dt, \quad (2.3) \]

with \( h = 2/\sigma \); see (1.4) and (1.5). Then the adjoint \( T_\sigma^* \) is given by

\[ (T_\sigma^* \hat{v})(t) = \frac{1}{\sqrt{h}} \sum_{j \geq 1} v_j \chi_{(j-1)h,jh]}(t) \quad (2.4) \]

where \( \hat{v}(z) = \sum_{j \geq 0} v_j z^j \in H^2(\mathbb{D}) \) and \( \chi_I(\cdot) \) denotes the characteristic function of the interval \( I \).

It is worth noticing that the operator \( T_\sigma : L^2(\mathbb{R}_+) \to H^2(\mathbb{D}) \) is a coisometry. This can be seen as follows:

\[ \|T_\sigma^* \hat{v}\|_{L^2(\mathbb{R}_+)}^2 = \frac{1}{h} \int_0^\infty |\sum_{j \geq 1} v_j \chi_{(j-1)h,jh]}|^2 \, dt = \frac{1}{h} \int_0^\infty \sum_{j \geq 1} |v_j|^2 \chi_{(j-1)h,jh]} \, dt \]

\[ = \frac{1}{h} \sum_{j \geq 1} |v_j|^2 \int_0^\infty \chi_{(j-1)h,jh]} \, dt = \sum_{j \geq 1} |v_j|^2 = \| \hat{v} \|_{H^2(\mathbb{D})}^2. \quad (2.5) \]

2.3 Approximation of the Laplace transform.

Let us now use the discrete time trajectories of (1.5) to approximate the continuous time dynamics in (1.3).

Let \( u \in L^2(\mathbb{R}_+) \) be arbitrary. In the operator notation, the output of the discretized dynamics (1.5) (after interpolation by \( T_\sigma^* \) back to a continuous time signal) is given by \( T_\sigma^* \mathbf{D}_\sigma T_\sigma u \). The output of continuous time dynamics (1.3) is given by \( \mathcal{L} \mathbf{G} \mathbf{L} u \). Our first task is to show that at least for some nice \( u \in L^2(\mathbb{R}_+) \) and \( T > 0 \) we have convergence

\[ \|T_\sigma^* \mathbf{D}_\sigma T_\sigma u - \mathcal{L} \mathbf{G} \mathbf{L} u\|_{L^2([0,T])} \to 0 \quad (2.6) \]
at some speed as \( \sigma \to \infty \). By Proposition 2 and equation (2.2) we see that
\[
T_{\sigma}^*D_{\sigma}T_{\sigma} = T_{\sigma}^*(C_{\sigma}M_{\sigma}^{-1}) \cdot G \cdot (M_{\sigma}C_{\sigma}^{-1}) T_{\sigma}
= T_{\sigma}^*(M_{\sigma}C_{\sigma}^{-1})^{-1} \cdot G \cdot (M_{\sigma}C_{\sigma}^{-1}) T_{\sigma} = (M_{\sigma}C_{\sigma}^{-1}T_{\sigma})^* \cdot G \cdot (M_{\sigma}C_{\sigma}^{-1}T_{\sigma})
\]
since the multiplication operator \( M_{\sigma} \) commutes with \( G \). Hence by (2.6), we are led to inquire whether the operators \( L_{\sigma} := M_{\sigma}C_{\sigma}^{-1}T_{\sigma} \) are close (on compact intervals) to the Laplace transform \( L \) when \( \sigma \) is large. This, indeed, appears to be true to some extent \(^3\).

**Proposition 3.** For any \( u \in C_c(\mathbb{R}_+) \) and \( s \in \mathbb{C}_+ \), we have \( (Lu)(s) = \lim_{\sigma \to \infty} (L_\sigma u)(s) \) where \( L_\sigma \) is defined as above.

**Proof.** Defining \( T_\sigma \) by (2.3) we get
\[
(L_\sigma u)(s) = \frac{\sqrt{2/\sigma}}{1 + s/\sigma} \sum_{j \geq 1} \left( \frac{1}{h} \int_{(j-1)h}^{jh} u(t) \, dt \right) \left( \frac{\sigma - s}{\sigma + s} \right)^j \tag{2.7}
\]
\[
= \frac{1}{1 + s/\sigma} \sum_{j \geq 1} \left( \int_0^\infty \chi_{[(j-1)h,jh]}(t) \left( \frac{\sigma - s}{\sigma + s} \right)^j u(t) \, dt \right)
= \int_0^\infty K_{s,\sigma}(t) u(t) \, dt,
\]
where \( \sigma = 2/h \) and
\[
K_{s,\sigma}(t) = \frac{1}{1 + s/\sigma} \sum_{j \geq 1} \chi_{[(j-1)h,jh]}(t) \left( 1 - \frac{2s}{s + \sigma} \right)^j. \tag{2.8}
\]
Now, if \( j \) is such that \( t \in [(j-1)h,jh] \), then we obtain from the previous
\[
K_{s,\sigma}(t) \approx \frac{1}{1 + s/\sigma} \left( 1 - \frac{s}{s/2 + \sigma/2} \right)^{(\sigma/2) \cdot t} \to e^{-st} \text{ as } \sigma \to \infty.
\]
We conclude that \( \lim_{\sigma \to \infty} K_{s,\sigma}(t) = e^{-st} \) for all \( s \in \mathbb{C}_+ \) and \( t \geq 0 \). Moreover, for each fixed \( s \in \mathbb{C}_+ \) and \( \sigma \geq 2|s| \) we have
\[
|K_{s,\sigma}(t)| \leq 2 \cdot \left( 1 + \frac{2|s|}{\sigma - |s|} \right)^{(\sigma/2) \cdot t} \leq 2 \cdot \left( 1 + \frac{2|s|}{\sigma - |s|} \right)^{|s| t/2} \cdot \left( 1 + \frac{2|s|}{\sigma - |s|} \right)^{|s| t/2} \leq 2 \left( e \sqrt{3} \right)^{|s| t}.
\]
The proposition now follows from the Lebesgue dominated convergence theorem, as the integrand in (2.7) is has a compact support. \( \square \)

The purpose of this paper is to give stronger versions of Proposition 3.

\(^3\)Note that by Proposition 2 and equality (2.5), we see that each \( L_\sigma : L^2(\mathbb{R}_+) \to H^2(\mathbb{C}_+) \) is a coisometry. The Laplace transform, in its turn, is an unitary mapping between the same spaces. Hence, the convergence of \( L_\sigma \to L \) must be rather weak.
3 A pointwise convergence estimate

Our main result will be given in this section. Theorem 1 provides a uniform speed estimate for the convergence of \((L_\sigma u)(i\omega) \to (Lu)(i\omega)\) for \(i\omega \in K\) where \(K \subset i\mathbb{R}\) is compact.

Before that some new definitions and notations must be given: Let \(I_j = ((j-1)h, jh]\) for some \(j \in \mathbb{N}\) and \(t_{j-1/2} = \frac{1}{2}(t_{j-1} + t_j)\). For \(u \in L^2(\mathbb{R}_+), \) let \(I_{h,s}u\) be the piecewise constant interpolating function, defined by

\[
(I_{h,s}u)(t) = \bar{u}_{j,h} + \frac{c_j(h,s)}{h}(t - t_{j-1/2}), \quad t \in I_j,
\]

where \(\bar{u}_{j,h} = \frac{1}{h} \int_{I_j} u(t) \, dt\) and the defining sequence \(\{c_j(h,s)\}_{j \geq 1}\) (depending on two parameters \(h\) and \(s\)) will be later chosen in a particular way. Let \(P_h\) denote the orthogonal projection in \(L^2(\mathbb{R}_+)\) onto the subspace of functions that are constant on each interval \(I_j\). Then clearly for all \(u \in L^2(\mathbb{R}_+), \) \(j \geq 1\) and \(t \in I_j\) we have \((P_hu)(t) = \bar{u}_{j,h}\).

**Theorem 1.** Let \(h > 0, \sigma = 2/h, \) \(T = Jh\) for some \(J \in \mathbb{N}, \) \(u \in C_c(\mathbb{R}_+) \cap H^1(\mathbb{R}_+), \) and assume that \(\text{supp}(u) := \{t \in \mathbb{R} : u(t) \neq 0\} \subset [0, T].\)

(i) Then the sequence \(\{c_j(h,s)\}_{j \geq 1}\) can be chosen so that \((L_\sigma - L)(I_{h,s}u)(s) = 0\) for all \(s \in \overline{\mathbb{R}_+}.\)

(ii) For any such choice of the sequence \(\{c_j(h,s)\}_{j \geq 1}\), we have

\[
|(L_\sigma u)(s) - (Lu)(s)| \leq \frac{hT^{1/2}|s|}{\pi} \left(\|I_{h,s}u - P_hu\|_{L^2([0,T])} + \frac{h}{\pi} |u|_{H^1([0,T])}\right)
\]

for all \(s \in \overline{\mathbb{R}_+}.\)

(iii) The sequence \(\{c_j(h,s)\}_{j \geq 1}\) in claim (i) can be chosen optimally so that

\[
\|I_{h,s}u - P_hu\|_{L^2([0,T])} \leq \frac{15}{218} \left(\frac{h^{-1/2}T^{-1/2}}{12} + \frac{|s|}{6e}\right) \|P_hu\|_{L^2([0,T])}
\]

for a given \(s \in i\mathbb{R}, \) \(T \geq 1\) if \(9h \leq T^{2/3}e^{-\frac{4}{3}|s|T}.\) Furthermore, then

\[
|(L_\sigma u)(s) - (Lu)(s)| \leq \frac{3h^{1/2}|s|}{100} \|u\|_{L^2([0,T])} + \frac{2hT^{1/2}|s|^2}{1000} \|u\|_{L^2([0,T])}
\]

\[
+ \frac{h^2 T^{1/2}|s|}{10} |u|_{H^1([0,T])}.
\]

**Proof.** Let us first make some general observations. By a simple argument,

\[
\|P_hu\|^2_{L^2(\mathbb{R}_+)} = h \sum_{j \geq 1} \bar{u}_{j,h}^2. \quad \text{Clearly for all } t \in I_j
\]

\[
(I_{h,s}u - P_hu)(t) = \frac{c_j(h,s)}{h}(t - t_{j-1/2}).
\]
Since for any $b > a$ we have
\[
\frac{1}{(b-a)^2} \int_a^b \left( t - \frac{b+a}{2} \right)^2 = \frac{b-a}{12},
\]
it follows that
\[
\|I_{h,s}u - P_h u\|_{L^2([0,T])}^2 = \sum_{j=1}^J \frac{c_j(h,s)^2}{h^2} \int_{t_{j-1}}^{t_j} (t - t_{j-1/2})^2 dt \tag{3.4}
\]
\[
= \frac{h}{12} \sum_{j=1}^J c_j(h,s)^2.
\]

In claim (i) we want to determine the sequence $\{c_j(h,s)\}_{j \geq 1}$ so as to satisfy
\[
(L_\sigma - \mathcal{L})(I_{h,s} u)(s) = 0 \text{ for given } h \text{ and } s.
\]
After some computations, we see that this is equivalent to requiring that $\{c_j(h,s)\}_{j \geq 1}$ satisfies
\[
\sum_{j=1}^J \bar{u}_{j,h} I_j^{(0)}(h,s) + \sum_{j=1}^J c_j(h,s) J_j(h,s) = 0, \tag{3.5}
\]
where for $s \in \mathbb{C}_+ \setminus \{0\}$
\[
I_j^{(0)}(h,s) := \int_{I_j} \left[ \frac{1}{1+s/\sigma} \left( \frac{\sigma - s}{\sigma + s} \right)^j e^{-st} \right] dt \tag{3.6}
\]
\[
= \frac{2}{\sigma + s} \left( \frac{\sigma - s}{\sigma + s} \right)^j + \frac{1}{s} \left[ e^{-sjh} - e^{-s(j-1)h} \right],
\]
and
\[
J_j(h,s) := I_j^{(1)}(h,s) - (j - 1/2)h \cdot I_j^{(0)}(h,s) \tag{3.7}
\]
\[
= \frac{1}{s^2} \left[ e^{-sjh} - e^{-s(j-1)h} \right] + \frac{h}{2s} \left[ e^{-sjh} + e^{-s(j-1)h} \right],
\]

Together with
\[
I_j^{(1)}(h,s) := \int_{I_j} \left[ \frac{1}{1+s/\sigma} \left( \frac{\sigma - s}{\sigma + s} \right)^j e^{-st} \right] t dt
\]
\[
= \frac{(2j-1)h}{\sigma + s} \left( \frac{\sigma - s}{\sigma + s} \right)^j + \left( \frac{jh + 1}{s} \right) \left[ e^{-sjh} - e^{-s(j-1)h} \right] + \frac{h}{s^2} e^{-s(j-1)h}.
\]

It is clear that (3.5) has a huge number of solutions $\{c_j(h,s)\}_{j \geq 1}$ for any fixed $s$ and $h$, and most of the functions $(h,s) \mapsto c_j(h,s)$ need not even be continuous.
Claim (ii) is to be treated next. Recalling (2.7), (2.8) and (3.1)

\[(L_\sigma u)(s) - (Lu)(s) = \int_0^T (K_{s,\sigma}(t) - e^{-st})u(t) \, dt\]

\[= \int_0^T (K_{s,\sigma}(t) - e^{-st})(u(t) - (I_{h,s}u)(t)) \, dt\]

\[= \sum_{j=1}^J \int_{t_{j-1}}^{t_j} (K_{s,\sigma}(t) - e^{-st})(u(t) - \bar{u}_{j,h}) \, dt\]

\[- \sum_{j=1}^J \frac{c_j(h, s)}{h} \int_{t_{j-1}}^{t_j} (K_{s,\sigma}(t) - e^{-st})(t - t_{j-1/2}) \, dt = I - II.\]  

Let us first give an estimate to the term II. By the Poincare inequality, Proposition 6, we obtain for all \(j = 1, \ldots, J\)

\[\|(I - P_h)(K_{s,\sigma} - e^{-s(\cdot)})\|_{L^2(I_j)} \leq \frac{h}{\pi} |K_{s,\sigma} - e^{-s(\cdot)}|_{H^1(I_j)} = \frac{h}{\pi} |e^{-s(\cdot)}|_{H^1(I_j)},\]

where the equality follows because the function \(K_{s,\sigma}\) is constant on each interval \(I_j\). By the mean value theorem we get for \(s \in \mathbb{C}_+\) and \(0 \leq a < b < \infty\),

\[|e^{-s(\cdot)}|_{H^1([a,b])} = \int_a^b \left| \frac{d}{dt} e^{-st}\right|^2 \, dt = \frac{|s|^2}{2Re s} (e^{-2aRe s} - e^{-2bRe s})\]

\[\leq \frac{|s|^2}{2Re s} \cdot 2Re s e^{-2Re s} (b - a) \leq (b - a)|s|^2 e^{-2Re s}.\]

Hence \(|e^{-s(\cdot)}|_{H^1(I_j)} \leq h^{1/2}|s|e^{-(j-1)hRe s}\) and this estimate is seen to hold also for all \(s \in \mathbb{C}_+\). We now conclude that \(|e^{-s(\cdot)}|_{H^1([0,T])} \leq T^{1/2}|s|\) and

\[\|(I - P_h)(K_{s,\sigma} - e^{-s(\cdot)})\|_{L^2(I_j)} \leq \frac{h^{3/2}|s|}{\pi}\]  

for all \(s \in \mathbb{C}_+\). Using (3.9) we have

\[II = \sum_{j=1}^J \int_{t_{j-1}}^{t_j} (K_{s,\sigma}(t) - e^{-st}) \cdot \frac{c_j(h, s)}{h} (t - t_{j-1/2}) \, dt\]  

\[= \sum_{j=1}^J \int_{t_{j-1}}^{t_j} ((I - P_h) (K_{s,\sigma} - e^{-s(\cdot)})(t) \cdot \frac{c_j(h, s)}{h} (t - t_{j-1/2}) \, dt\]

\[\leq \sum_{j=1}^J \frac{h^{3/2}|s|}{\pi} \cdot \left[ \frac{c_j(h, s)^2}{h^2} \int_{t_{j-1}}^{t_j} (t - t_{j-1/2})^2 \, dt \right]^{1/2}\]

\[\leq \left( \sum_{j=1}^J \frac{h^{3/2}|s|^2}{\pi^2} \right)^{1/2} \cdot \left( \sum_{j=1}^J \frac{c_j(h, s)^2}{h^2} \int_{t_{j-1}}^{t_j} (t - t_{j-1/2})^2 \, dt \right)^{1/2}\]

\[\leq \frac{h^{3/2}|s|}{\pi} J^{1/2} \cdot \|I_{h,s}u - P_h u\|_{L^2([0,T])} = \frac{hT^{1/2}|s|}{\pi} \|I_{h,s}u - P_h u\|_{L^2([0,T])}\]
where the Schwarz inequality has been used twice, and the second to last step is by (3.4).

It remains to estimate term I in (3.8). In this case, since \( P_h \) maps on piecewise constant functions and each \( u(t) - \tilde{u}_{j,h} \) has zero mean on subintervals \( I_j \), we obtain by the inequalities of Schwarz and Poincare, together with (3.9)

\[
II \leq \sum_{j=1}^{J} \int_{I_{j-1}}^{t_j} ((I - P_h) \left( K_{s,a} - e^{-s(\cdot)} \right) (t))(u(t) - \tilde{u}_{j,h}) \, dt
\]

\[
\leq \sum_{j=1}^{J} \frac{h^{3/2}|s|}{\pi} \cdot \frac{h}{\pi} |u|_{H^1(I_j)} \leq \frac{h^{5/2}|s|}{\pi^2} \sum_{j=1}^{J} |u|_{H^1(I_j)} \tag{3.11}
\]

\[
\leq \frac{h^{5/2}|s|}{\pi^2} \left( \sum_{j=1}^{J} 1 \right)^{1/2} \left( \sum_{j=1}^{J} |u|_{H^1(I_j)}^2 \right)^{1/2} = \frac{h^2 T^{1/2}|s|}{\pi^2} |u|_{H^1([0,T])}.
\]

Estimate (3.2) follows from combining (3.10) and (3.11) with (3.8).

To prove claim (iii), we shall minimise \( \frac{h}{12} \sum_{j=1}^{J} c_j(h,s)^2 \) under the constraint (3.5), see (3.4) for motivation. We form the Langrange function

\[
L(c_1, \ldots, c_J, \lambda) = \frac{h}{12} \sum_{j=1}^{J} c_j^2 + \lambda \left( \sum_{j=1}^{J} \tilde{u}_{j,h} I_j^{(0)}(h,s) + \sum_{j=1}^{J} c_j J_j(h,s) \right),
\]

and compute its (unique) critical point giving the minimum. We obtain

\[
\left\{ \begin{array}{l}
\frac{\partial L}{\partial c_k} = \frac{h}{6} c_k + \lambda J_k(h,s) = 0 \quad \text{for } 1 \leq k \leq J, \\
\sum_{j=1}^{J} \tilde{u}_{j,h} I_j^{(0)}(h,s) + \sum_{j=1}^{J} c_j J_j(h,s) = 0.
\end{array} \right.
\]

Solving this gives the minimising sequence

\[
c_k = c_k(h,s) = -\frac{6\lambda}{h} J_k(h,s) = -\frac{\sum_{j=1}^{J} \tilde{u}_{j,h} I_j^{(0)}(h,s)}{\sum_{j=1}^{J} J_j(h,s)^2} J_k(h,s),
\]

for all \( 1 \leq k \leq J \), and then for the minimum value

\[
\frac{h}{12} \sum_{j=1}^{J} c_j(h,s)^2 = \frac{h}{12} \left( \frac{\sum_{j=1}^{J} \tilde{u}_{j,h} I_j^{(0)}(h,s)}{\sum_{j=1}^{J} J_j(h,s)^2} \right)^2 \sum_{k=1}^{J} J_k(h,s)^2
\]

\[
= \frac{h}{12} \frac{\left( \sum_{j=1}^{J} \tilde{u}_{j,h} I_j^{(0)}(h,s) \right)^2}{\sum_{j=1}^{J} J_j(h,s)^2}.
\]

Hence, choosing the operator \( I_{h,s} \) in (3.4) optimally gives

\[
\|I_{h,s} u - P_h u\|_{L^2([0,T])} \leq \frac{\left( \sum_{j=1}^{J} I_j^{(0)}(h,s)^2 \right)^{1/2}}{\left( \sum_{j=1}^{J} J_j(h,s)^2 \right)^{1/2}} \frac{\|P_h u\|_{L^2([0,T])}}{2\sqrt{3}}
\]
since \( \|P_h u\|_{L^2([0,T])} = \left( h \sum_{j=1}^{\infty} \bar{u}_{j,h}^2 \right)^{1/2} \) . We must now attack (3.6) and (3.7) to estimate the required two square sums, and the resulting long computations will be done in separate subsections 3.1 and 3.2. As a final result, we get by Propositions 4 and 5

\[
\left( \sum_{j=1}^{J} J_j(0) (h, s)^2 \right)^{1/2} \leq \frac{5}{218} \left( 3h^{-1/2}T^{-1/2} + h^{1/2}|s|^2T^{1/2} \right)
\]

assuming that \( 9h \leq T^{2/3}e^{-\frac{3}{4}|s|T} \). But then

\[
h^{1/2}|s|^2T^{1/2} \leq \frac{|s|}{3} \cdot |s|T^{5/6}e^{-\frac{3}{4}|s|T} \leq \frac{|s|}{3} \cdot |s|T e^{-\frac{3}{4}|s|T} \leq \frac{|s|}{2e} .
\]

since \( \max_{r \geq 0} re^{-\frac{3}{4}r} = 3/(2e) \). Noting that the norm of the orthogonal projection \( P_h \) is 1, the proof of 1 is now complete.

3.1 Estimation of (3.7)

In this subsection, we shall estimate the square sum of

\[
J_j(h, s) = \frac{1}{s^2} \left[ e^{-sjh} - e^{-s(j-1)h} \right] + \frac{h}{2s} \left[ e^{-sjh} + e^{-s(j-1)h} \right]
\]

(3.12)

from below and above. For the first term on the left of (3.12) we obtain

\[
\frac{1}{s^2} \left[ e^{-sjh} - e^{-s(j-1)h} \right] = \frac{1}{s^2} \left[ \sum_{k \geq 0} \frac{(-sjh)^k}{k!} - \sum_{k \geq 0} \frac{(-s(j-1)h)^k}{k!} \right]
\]

\[
= \frac{1}{s^2} \left[ -sh + \sum_{k \geq 2} \frac{(-sh)^k(j^k - (j-1)^k)}{k!} \right]
\]

\[
= -\frac{h}{s} + \sum_{k \geq 2} \frac{(j^k - (j-1)^k)}{k!}(-s)^{k-2}h^k.
\]

For the latter term in (3.12) we get

\[
\frac{h}{2s} \left[ e^{-sjh} + e^{-s(j-1)h} \right] = \frac{h}{s} \sum_{k \geq 0} \frac{(-s)^k(j^k + (j-1)^k)}{2k!} h^k
\]

\[
= \frac{h}{s} - \sum_{k \geq 2} \frac{(j^{k-1} + (j-1)^{k-1})}{2(k-1)!}(-s)^{k-2}h^k.
\]

Hence, for all \( s \in \mathbb{C}_+ \setminus \{0\} \)

\[
J_j(h, s) = \sum_{k \geq 2} \frac{d_k(j)}{2k!}(-s)^{k-2}h^k
\]
Finally, we get
\[ d_k(j) = 2(j^k - (j - 1)^k) - k(j^{k-1} + (j - 1)^{k-1}) \]
\[ = \sum_{m=0}^{k-3} \binom{k}{m} (k - m - 2)(-1)^{k-m}j^m \quad \text{for} \quad k \geq 3 \]
and \( d_2(j) = 0 \). Hence \( d_k(j) \) is a polynomial of degree \( k - 3 \) in variable \( j \).
Finally, we get
\[ J_j(h, s) = \sum_{k \geq 3} \sum_{m=0}^{k-3} \frac{k - m - 2}{2m!(k - m)!} (-j)^m s^{k-2} h^k. \]
Let us compute an upper estimate for
\[ \| \{J_j(h, s)\}_j\|_{\ell^2} := \left( \sum_{j=1}^{J} J_j(h, s) \right)^{1/2}. \]
By the triangle inequality
\[ \| \{J_j(h, s)\}_j\|_{\ell^2} \]
\[ \leq |s|^{-2} \cdot \sum_{k \geq 3} \sum_{m=0}^{k-3} \frac{k - m - 2}{2m!(k - m)!} |sh|^k \left( \sum_{j=1}^{J} j^{2m} \right)^{1/2} \]
\[ \leq |s|^{-2} \cdot \sum_{k \geq 3} \sum_{m=0}^{k-3} \frac{k - m - 2}{2m!(k - m)!} |sh|^k \cdot \frac{J^{m+1/2}}{\sqrt{2m + 1}} \]
\[ \leq \frac{1}{2}|s|T^{1/2}h^{5/2} \cdot \sum_{k \geq 3} \sum_{m=0}^{k-3} \frac{k - m - 2}{2\sqrt{2m + 1} m!(k - m)!} |s|^{k-3} T^m h^{k-m-3}. \]
Noting that for \( k - 3 \geq m \geq 0 \) we have \( \frac{k-m-2}{\sqrt{2m + 1} m!(k - m)!} \leq \frac{1}{m!(k-m-3)!} \) and \( |s|^{k-3} T^m h^{k-m-3} = |sh|^{k-3} \cdot (T/h)^m \), we may estimate the sum term above
\[ \sum_{k \geq 3} \sum_{m=0}^{k-3} \frac{k - m - 2}{2\sqrt{2m + 1} m!(k - m)!} |s|^{k-3} T^m h^{k-m-3} \]
\[ \leq \sum_{k \geq 3} \frac{|sh|^{k-3}}{(k-3)!} \sum_{m=0}^{k-3} \left( \frac{k-3}{m} \right) \left( \frac{T}{h} \right)^m \]
\[ \leq \sum_{k \geq 3} \frac{|sh|^{k-3}}{(k-3)!} \left( 1 + \frac{T}{h} \right)^{k-3} = e^{|s|(h+T)}. \]
We now conclude for all \( h, T > 0 \) and \( s \in \mathbb{C}_+ \setminus \{0\} \) that
\[ \| \{J_j(h, s)\}_j\|_{\ell^2} \leq \frac{1}{2}|s|T^{1/2}h^{5/2}e^{|s|(h+T)}. \quad (3.13) \]
In addition to estimate (3.13) a lower bound can also be obtained: Decompose

\[ J_j(h, s) = \sum_{k=3}^{\infty} \sum_{m=0}^{k-3} \frac{k-m-2}{2m!(k-m)!} (-j)^m s^{k-2} h^k \]

\[ = \sum_{k=3}^{\infty} \frac{1}{2(k-3)!3!} (-j)^{k-3} s^{k-2} h^k + \sum_{m=0}^{k-4} \frac{k-m-2}{2m!(k-m)!} (-j)^m s^{k-2} h^k \]

so that by the triangle inequality

\[
\| J_j(h, s) \|_{L^2} \geq \left\| \sum_{k=3}^{\infty} \frac{1}{2(k-3)!3!} (-j)^{k-3} s^{k-2} h^k \right\|_{L^2} \\
- \left\| \sum_{k=4}^{\infty} \sum_{m=0}^{k-4} \frac{k-m-2}{2m!(k-m)!} (-j)^m s^{k-2} h^k \right\|_{L^2}.
\]

For the first term in the right hand side of (3.14) we have

\[
\left\| \sum_{k=3}^{\infty} \frac{1}{2(k-3)!3!} (-j)^{k-3} s^{k-2} h^k \right\|_{L^2} \geq 0
\]

\[ = \left\| \frac{1}{12} h^3 \sum_{k=3}^{\infty} \frac{1}{(k-3)!} (-j)^{k-3} s^{k-3} h^{k-3} \right\|_{L^2} \]

where

\[
\left\| \left\{ e^{-jsh} \right\}_{j=1}^{J} \right\|_{L^2} = \sum_{j=1}^{J} |e^{-jsh}|^2
\]

\[
= \begin{cases} 
J = h^{-1} T, & \text{when } \text{Re} \ s = 0 \\
\frac{e^{-2h\text{Re} \ s} 1 - e^{-2(J+1)h\text{Re} \ s}}{1 - e^{-2h\text{Re} \ s}}, & \text{when } \text{Re} \ s > 0.
\end{cases}
\]
Indeed, in this subsection, we compute an upper estimate for Proposition 4. Since always \( T > 0 \), we have a similar upper estimate to (3.13). As a conclusion we can now state

**Proposition 4.** Let \( J_j(h, s) \) be defined through (3.12). Then for any \( s \in \mathbb{R} \), \( T, h > 0 \) satisfying \( T = Jh \), \( J \in \mathbb{N} \) and \( 9h \leq T^{2/3}e^{-\frac{1}{3}|s|T} \) we have

\[
\| \{ J_j(h, s) \}_{j=1}^J \|_2 \geq \frac{5}{109} |T h^2| s | \tag{3.18}
\]

**Proof.** It is clear that (3.18) is satisfied for \( s = 0 \). For \( s \in \mathbb{R} \setminus \{0\} \) it follows from (3.14) and (3.15) – (3.17) that for all \( s \in \mathbb{R} \setminus \{0\} \), \( h, T > 0 \) satisfying \( T = Jh \) for \( J \in \mathbb{N} \) that the estimate

\[
\| \{ J_j(h, s) \}_{j=1}^J \|_2 \geq \left( \frac{T}{12} - h^{3/2} e^{\frac{1}{3}|s| (h + T)} \right) h^2 |s|
\]

holds. Since always \( h \leq T \), we have \( h^{3/2} e^{\frac{1}{3}|s| (h + T)} \leq h^{3/2} e^{2|s| T} \leq \frac{T}{27} \) provided that \( h \leq \frac{T^{3/3}}{9} e^{-\frac{1}{3}|s| T} \). The claim follows from this. \( \square \)

### 3.2 Estimation of (3.6)

In this subsection, we compute an upper estimate for

\[
\| \{ I_j^{(0)}(h, s) \}_{j=1}^J \|_2 := \left( \sum_{j=1}^J I_j^{(0)}(h, s)^2 \right)^{1/2}
\]

Writing \( \tau = sh \) and recalling \( \sigma = 2/h \), we get for \( s \in \mathbb{C}_+ \)

\[
I_j^{(0)}(h, s) = \frac{2}{\sigma + s} \left( \frac{\sigma - s}{\sigma + s} \right)^j + \frac{1}{s} \left( e^{-sjh} - e^{-s(j-1)h} \right)
\]

\[
= \frac{2}{\sigma + s} \left( \frac{\sigma - s}{\sigma + s} \right)^j - e^{-sjh} + \left( \frac{2}{\sigma + s} - \frac{1}{s} (e^{sh} - 1) \right) e^{-sjh}
\]

\[
= \frac{2h}{2 + \tau} \left( \frac{2 - \tau}{2 + \tau} \right)^j - e^{-\frac{h}{\tau}} + \left( \frac{2h}{2 + \tau} - \frac{h}{\tau} (e^{\tau} - 1) \right) e^{-\frac{h}{\tau}}.
\]
Let $\Omega \subset \overline{C}_+$ be any set. Then for any $\tau \in \Omega$ we have

$$\left| I_j^{(0)}(h, s) \right| \leq \frac{2h}{2 + \tau} \left| \left( \frac{2 - \tau}{2 + \tau} \right)^j - e^{-\tau j} \right| + \left| \frac{2h}{2 + \tau} - \frac{h}{\tau} (e^\tau - 1) \right| e^{-\tau j}$$

$$\leq \frac{2h}{2 + \tau} \left| \left( \frac{2 - \tau}{2 + \tau} \right) - e^{-\tau} \right| \sum_{k=1}^{j-1} \left( \frac{2 - \tau}{2 + \tau} \right)^k e^{-\tau(j-k-1)}$$

$$+ \left| \frac{2h}{2 + \tau} - \frac{h}{\tau} (e^\tau - 1) \right|$$

$$\leq h|\tau| \left( C_\Omega \left| \frac{2j\tau^2}{2 + \tau} \right| + C'_\Omega \right)$$

where the constants are given by

$$C_\Omega = \sup_{\tau \in \Omega} \left| \frac{1}{\tau^3} \left( \frac{2 - \tau}{2 + \tau} - e^{-\tau} \right) \right|$$

and

$$C'_\Omega = \sup_{\tau \in \Omega} \left| \frac{1}{\tau} \left( \frac{2}{2 + \tau} - \frac{1}{\tau} (e^\tau - 1) \right) \right|.$$
Similarly
\[ C'_\Omega \leq \sum_{j \geq 0} \left| \left( \frac{-1}{2} \right)^{j+1} - \frac{1}{(j+2)!} \right| \cdot \left( \frac{1}{12e} \right)^j \leq \sum_{j \geq 0} \frac{1}{2^j} \cdot \left( \frac{1}{12e} \right)^j \]
\[ = \frac{24e}{24e - 1} < \frac{3}{2} \]
But now (3.19) implies (3.20). □

### 3.3 Determination of the isoperimetric constant

In this section we give a basic interpolation estimate used several times in the proofs.

**Proposition 6.** Assume that \( u \in H^1(I_j) \). Then
\[
\|u - \bar{u}\|_{L^2(I_j)} \leq \frac{h}{\pi} |u|_{H^1(I_j)}
\]

**Proof.** Let \( I_{ref} = (0, 1] \) and define the bilinear forms
\[
a(u, v) = \int_{I_{ref}} u'(v')^* \, dt
\]
and
\[
b(u, v) = \int_{I_{ref}} uv^\ast \, dt
\]
where the asterisk denotes complex conjugation. Furthermore, let
\[
V = \{ v \in H^1(I_{ref}) \mid \int_{I_{ref}} v(t) \, dt = 0 \}
\]
and
\[
\lambda_1 = \inf_{v \in V, v \neq 0} \frac{a(v, v)}{b(v, v)} \in \mathbb{R}^+
\]
By Rayleigh’s theorem, \( \lambda_1 \) is the smallest eigenvalue of the problem: Find \( u \in V \) such that
\[
a(u, v) = \lambda b(u, v) \quad \forall v \in V.
\]
(3.21)
Solution to (3.21) can be sought for using the Euler equations for the eigenpair \((\lambda, u)\). By standard calculus the first eigenpair is found to be \((\lambda_1, u_1) = (\pi^2, \cos(\pi t))\). It follows that \( b(v, v) \leq \frac{1}{\lambda_1} a(v, v) \), that is \( \|v\|^2_{L^2(I_{ref})} \leq \frac{1}{\pi^2} |v|^2_{H^1(I_{ref})} \) for any \( v \in V \). Let now \( u \in H^1(I_{ref}) \) and set \( v = u - \bar{u} \in V \) implying
\[
\|u - \bar{u}\|^2_{L^2(I_{ref})} \leq \frac{1}{\pi^2} |u - \bar{u}|^2_{H^1(I_{ref})} = \frac{1}{\pi^2} |u|^2_{H^1(I_{ref})}
\]
(3.22)
For the general interval \( I_j = (t_{j-1}, t_j] \) a standard scaling argument with \( \bar{u}(\tau) = u((t - t_{j-1})/h) \) and \( \tau = (t - t_{j-1})/h \in I_{ref} \) gives
\[
\|u - \bar{u}\|^2_{L^2(I_j)} = h \|\hat{u} - \bar{u}\|^2_{L^2(I_{ref})} \leq \frac{1}{\pi^2} h \|u\|^2_{H^1(I_{ref})} = \frac{1}{\pi^2} h^2 |u|^2_{H^1(I_j)}
\]
(3.23)
implying
\[
\|u - \bar{u}\|_{L^2(I_j)} \leq \frac{1}{\pi} h |u|_{H^1(I_j)}.
\]
(3.24)
□
4 Weak and strong convergence

We first show that Theorem 1 implies that $L_\sigma \to \mathcal{L}$ in weak operator topology. Using this, it is then shown in Theorem 2 that the convergence is, in fact, strong.

Indeed, it follows from Theorem 1 that $(L_\sigma u)(i\omega) \to (\mathcal{L}u)(i\omega)$ uniformly in the compact subsets $i\omega \in K \subset i\mathbb{R}$ for any $u \in C_c(\mathbb{R}_+) \cap H^1(\mathbb{R}_+)$. Hence, for finite linear combinations $s$ (also called simple functions) of characteristic functions $\chi_K$ of compact intervals $K \subset i\mathbb{R}$ we have $(s, L_\sigma u)_{L^2(i\mathbb{R})} \to (s, \mathcal{L}u)_{L^2(i\mathbb{R})}$. Since $\|L_\sigma\|_{(L^2(\mathbb{R}_+), H^2(\mathbb{C}_+))} \leq 1$ and simple functions are dense in $L^2(i\mathbb{R})$, it follows that

$$\langle v, L_\sigma u \rangle_{K^2(i\mathbb{R})} \to \langle v, \mathcal{L}u \rangle_{H^2(i\mathbb{R})} \text{ as } \sigma \to \infty$$

(4.1)

for all $u \in C_c(\mathbb{R}) \cap H^1(\mathbb{R}_+)$ and $v \in L^2(i\mathbb{R}_+)$. Another density argument implies finally that (4.1) holds even for all $u \in L^2(\mathbb{R}_+)$ and $v \in L^2(i\mathbb{R}_+)$.

We recall a result from elementary functional analysis:

**Proposition 7.** Let $H$ be a Hilbert space, and assume that $u_j \to u$ weakly in $H$. If $\|u_j\|_H \to \|u\|_H$, then $u_j \to u$ in the norm of $H$.

**Proof.**

$$\langle u_j - u, u_j - u \rangle_H = \langle u_j, u_j \rangle_H - \langle u, u \rangle_H - \langle u, u_j - u \rangle_H - \langle u_j - u, u \rangle_H = \|u_j\|^2_H - \|u\|^2_H - 2\text{Re} \langle u, u_j - u \rangle_H. \quad \square$$

**Theorem 2.** We have $\|L_\sigma u - \mathcal{L}u\|_{H^2(\mathbb{C}_+)} \to 0$ for any $u \in L^2(\mathbb{R}_+)$. Moreover, $\|L_\sigma^* v - \mathcal{L}^* v\|_{L^2(\mathbb{R}_+)} \to 0$ for any $v \in H^2(\mathbb{C}_+)$.  

**Proof.** Adjoining (4.1) shows that $L_\sigma^* v \to \mathcal{L}^* v$ weakly. Since $L_\sigma$ is a coisometry by Proposition 2 and (2.5), we have

$$\|L_\sigma^* v\|^2_{L^2(\mathbb{R}_+)} = \langle L_\sigma L_\sigma^* v, v \rangle_{H^2(\mathbb{C}_+)} = \|v\|^2_{H^2(\mathbb{C}_+)}. \quad \text{Now Proposition 7 implies the latter part of this Theorem.}$$

To show the first part, we have to work a bit harder to verify that $\|L_\sigma u\|_{L^2(i\mathbb{R})} \to \|u\|_{L^2(\mathbb{R}_+)} = \|\mathcal{L}u\|_{L^2(i\mathbb{R})}$. Suppose that $h = 2/\sigma > 0$ and $u \in L^2(\mathbb{R}_+)$ is such that $u(t) = \pi_{j,h} := \int_{((j-1)h,jh)} u(t) \, dt$ for all $t \in I_j := ((j-1)h,jh]$ — in other words, this is simply $u = P_h u$. For such $u$

$$\|u\|^2_{L^2(\mathbb{R}_+)} = \sum_{j \geq 1} \int_{I_j} |u(t)|^2 \, dt = h \|\{\pi_{j,h}\}_{j \geq 0}\|^2_{l^2}.$$

By the definition of the discretizing operator $T$, we have

$$\|T u\|^2_{H^2(\mathbb{D})} = \sum_{j \geq 1} \left( \frac{1}{\sqrt{h}} \int_{I_j} |u(t)|^2 \, dt \right)^2 = h \sum_{j \geq 1} |\pi_{j,h}|^2 = \|u\|^2_{L^2(\mathbb{R}_+)}.$$

Hence, we have $\|T_{\sigma} P_h u\|_{H^2(\mathbb{D})} = \|P_h u\|_{L^2(\mathbb{R}_+)}$ for all $u \in L^2(\mathbb{R}_+)$ where $\sigma = 2/h$. Also note that $T_{\sigma} u = T_{\sigma} P_h u$ for all $u \in L^2(\mathbb{R}_+)$ provided that $\sigma = 2/h$. 

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We now have for any $u \in L^2(\mathbb{R}_+)$
\[
||T_\sigma u||_{H^2(\mathbb{R}_+)} - ||u||_{L^2(\mathbb{R}_+)} ||
\leq ||T_\sigma u||_{H^2(\mathbb{R}_+)} - ||T_\sigma P_h u||_{H^2(\mathbb{R}_+)} + ||T_\sigma P_h u||_{H^2(\mathbb{R}_+)} - ||P_h u||_{L^2(\mathbb{R}_+)} + ||P_h u||_{L^2(\mathbb{R}_+)} - ||u||_{L^2(\mathbb{R}_+)}
\]
where again $\sigma = 2/h$. Since the projections $P_h \to I$ strongly in $L^2(\mathbb{R}_+)$ as $h \to 0$, we conclude that $||T_\sigma u||_{H^2(\mathbb{R}_+)} \to ||u||_{L^2(\mathbb{R}_+)}$ and hence $||L_\sigma u||_{H^2(\mathbb{R}_+)} \to ||u||_{L^2(\mathbb{R}_+)}$ as $\sigma \to \infty$, see Proposition 2. The first claim of this theorem follows from this, Proposition 7 and (4.1).

Using Theorem 2 we can show that the output of integration scheme (1.5) converges to the output of continuous time dynamics (1.3) for input/output stable systems $S$. These are systems for which $G(\cdot) \in H^\infty(\mathbb{C}_+)$ or, equivalently, $G \in \mathcal{L}(H^2(\mathbb{C}_+))$. To understand the formulation of the following theorem, we refer back to Section 2.

**Theorem 3.** For any $u \in L^2(\mathbb{R}_+)$ and $G \in H^\infty(\mathbb{C}_+)$, we have
\[
||T_\sigma^* D_\sigma T_\sigma u - L^* G \mathcal{L} u||_{L^2(\mathbb{R}_+)} \to 0
\] (4.2)
as $\sigma \to \infty$.

**Proof.** As noted just before Proposition 3, we have $T_\sigma^* D_\sigma T_\sigma = L_\sigma^* G \mathcal{L}$. Then we get for all $\sigma > 0$
\[
||L_\sigma^* G \mathcal{L} u - L^* G \mathcal{L} u||_{L^2(\mathbb{R}_+)} \leq ||(L_\sigma^* - L^*) G (L_\sigma u - \mathcal{L} u)||_{L^2(\mathbb{R}_+)} + ||(L_\sigma^* - L^*) G \mathcal{L} u||_{L^2(\mathbb{R}_+)} + ||L^* G (L_\sigma u - \mathcal{L} u)||_{L^2(\mathbb{R}_+)}.\]

Now (4.2) follows by Theorem 2. \hfill \Box

## 5 A counterexample

We complete this paper by reviewing estimate (2.6) in the special case when $G(s) = I$ for all $s \in \mathbb{C}_+$. It indicates that Theorem 3 cannot be improved by a speed estimate for convergence.

In this special case it follows from the very definitions that $L_\sigma^* G \mathcal{L} \sigma = T_\sigma^* T_\sigma = P_{2/\sigma}$ where the orthogonal projection $P_h$ is defined as in Section 3. Since $L^* \mathcal{L} = I$ on all of $L^2(\mathbb{R}_+)$, we should give an estimate to
\[
||u - P_h u||_{L^2([0,T])} \quad \text{for a family of functions} \quad u \in L^2(\mathbb{R}_+).
\]

It is, of course, true that $P_h u \to u$ as $h \to 0$ for all $u \in L^2(\mathbb{R}_+)$. However, there cannot be a uniform speed estimate of type
\[
||u - P_h u||_{L^2([0,T])} \leq C_\alpha h^\alpha
\] (5.1)
where $C_u < \infty$ for all $u \in L^2([0,T])$. If it were so, then for any $0 < \beta < \alpha$ we would have $\|h^{-\beta}(I - P_h)u\|_{L^2([0,T])} \leq C_u h^{\alpha - \beta} \to 0$ as $h \to 0$, for all $u \in L^2([0,T])$. By the uniform boundedness principle,

$$\sup_{h>0} \|h^{-\beta}(I - P_h)\|_{L^2([0,T])} =: M < \infty$$

and hence $\|(I - P_h)\|_{L(L^2([0,T]))} \leq M h^\beta$ for all $h > 0$.

Making now $h$ small enough, we see that then the norm of the orthogonal projection $(I - P_h)|L^2([0,T])$ is strictly less than 1; this implies that $I|L^2([0,T]) = P_h|L^2([0,T])$. But $P_h|L^2([0,T])$ is a finite rank operator, and the uniform speed estimate (5.1) cannot hold by contradiction. The same conclusion holds, if $h^\alpha$ in (5.1) is replaced by any increasing continuous function $\phi(h)$ satisfying $\phi(0) = 0$.

It should also be noted that for functions $u \in L^2(\mathbb{R}_+)$ that possess certain smoothness properties such a speed estimate can be obtained. See [2] for a further discussion on what is obtainable and what is not.

### 6 Conclusions

The operators $L_\sigma$ for $\sigma > 0$ have been introduced just before Proposition 3 with aid of the Cayley transformation (1.7). It is shown in Theorem 2 that the operators $L_\sigma$ provide an approximation to Laplace transform for a wide class of functions. In addition, Theorem 3 shows that for I/O-stable linear systems, the convergence extends to the input/output relation of the system. All this can be anticipated since the Cayley transform actually corresponds to the slightly “unorthodox”, conservativity-preserving discretization (1.5) of the dynamical equations (1.3) (or their infinite-dimensional analogue e.g. in [8, Proposition 2.5]).

Theorem 3 gives no estimate on the speed of the convergence with respect to the sampling parameter $h = 2/\sigma$. If we had some decay

$$G(s) \to 0 \quad \text{as} \quad |s| \to \infty$$

at some speed, then we could effectively restrict our analysis to compact subsets of $i\mathbb{R}$. Then the speed estimate of Theorem 1 could show up in (4.2) in some form. Unfortunately, (6.1) is not a generic property of $G \in H^\infty(\mathbb{C}_+)$ – hence it is not a generic property of the transfer functions of conservative systems either.

In the time domain, the same problem appears because the sampling operator $T_\sigma$ cannot detect above a certain cutoff frequency: there are always high-frequency signals carrying substantial energy that a given discretized system cannot capture. To achieve a speed estimate in (4.2), one could assume either

(i) that the high frequencies are damped by the linear system itself (e.g. by a property like (6.1)), or
(ii) that the high frequencies have a small amplitude in the signal $u$ (e.g. an assumption such as $u \in H^1(\mathbb{R}_+)$ in Theorem 1).

The approximation of the state trajectory $x(\cdot)$ by the discrete trajectories $\{x_j^{(h)}\}_{j \geq 0}$ solving (1.5) has not been studied here. This will be carried out in a future paper on the state space approximation for conservative systems.

**Remark 1.** We remark that practically all of the results presented in this paper hold if the input space of the node $S$ is a separable Hilbert space instead of $\mathbb{C}$.

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