FUNCTION HOPF ALGEBRA AND PSEUDODIFFERENTIAL OPERATORS ON COMPACT LIE GROUPS

Ville Turunen
FUNCTION HOPF ALGEBRA AND PSEUDODIFFERENTIAL OPERATORS ON COMPACT LIE GROUPS

Ville Turunen
**Ville Turunen:**  Function Hopf algebra and pseudodifferential operators on compact Lie groups; Helsinki University of Technology Institute of Mathematics Research Reports A465 (2004).

**Abstract:** The Schwartz kernel representation $(A \mapsto K_A): \mathcal{L}(\mathcal{H}) \to \hat{\mathcal{H}} \otimes \mathcal{H}^\prime$ endows the space of continuous linear operators on a nuclear Hopf-Fréchet algebra $\mathcal{H}$ with a natural Hopf algebra structure. We study pseudodifferential symbolic calculus on a compact Lie group $G$ related to the function Hopf algebra $\mathcal{H} := \mathcal{D}(G)$.

**AMS subject classifications:** 16W30, 47L80, 47G30, 22E30, 46E25, 43A77, 58J40, 35S05.

**Keywords:** Hopf algebras, pseudodifferential operators, compact Lie groups, function algebra, convolution.

Ville.Turunen@hut.fi

ISBN 951-22-6900-7
ISSN 0784-3143

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/
1 Introduction

Hopf algebras were introduced by Heinz Hopf in 1941, in the context of algebraic topology. Somewhat simplified, a Hopf algebra is an algebra for which the dual space is also an algebra so that the duality pairing is intertwined in a subtle symmetric way. Examples range from group algebras, their duals, and universal enveloping algebras to deformations of such structures.

The study of Hopf algebras can be considered as a kind of non-commutative geometry, stemming out from the 19th century observations of polynomial rings. Indeed, by the work of Hilbert and others, many commutative rings were realized as function algebras — recall the Gelfand theory of commutative Banach algebras culminating this line of thought in early 1940s: For a commutative unital $C^*$-algebra $A$, let $X := \text{Hom}(A, \mathbb{C})$. The Gelfand transform of $f \in A$ is $\hat{f} : X \to \mathbb{C}$ defined by $\hat{f}(x) := x(f)$. If $X$ is endowed with the weakest topology such that $\hat{A} := \{\hat{f} \mid f \in A\} \subset C(X)$ then it is a compact Hausdorff space and $C(X) = \hat{A} \cong A$. That is, $A$ is essentially an algebra of functions.

Hence, commutative algebra is closely related to geometry: instead of a space, we can study the function algebra on it. Non-commutative geometry is a concept referring to the study of not necessarily commutative algebras. We may associate $C^*$-algebras to non-commutative topology (commutative case: $C(X)$), von Neumann algebras to non-commutative measure theory (commutative case: $L^\infty(X)$), and Lipschitz-algebras to non-commutative metric theory (commutative case: $\text{Lip}(X)$) (see [7]).

What if Hom$(A, \mathbb{C})$ of a commutative $C^*$-algebra $A$ has a structure of a topological group? The group axioms give rise to operations on the algebra, and these new operations have natural symmetries reflected in the algebra; generalizing this to the non-commutative case, Hopf algebras arise. Nevertheless, Hopf algebras provide a satisfying duality theory for algebraic structures much more general than just groups. Purely algebraic Hopf theory is often spiced up with topology, and there are Hopf-von Neumann algebras, Hopf $C^*$-algebras, Hopf-Fréchet algebras, etc.

In the sequel, we study Hopf algebras inspired by symbolic calculus of pseudodifferential operators on a compact Lie group $G$. There the Hopf algebra $\mathcal{H}$ is the nuclear Fréchet algebra $\mathcal{D}(G)$ of functions $f \in C^\infty(G)$; the pseudodifferential operators that map $\mathcal{H}$ to $\mathcal{H}$ form a subalgebra of $\mathcal{L}(\mathcal{H})$. Instead of studying an operator $A \in \mathcal{L}(\mathcal{H})$, we study its symbol $\sigma_A : G \to \mathcal{L}(\mathcal{H})$, in some sense a less complicated object. The symbol of a pseudodifferential operator composition is approximately the product of the symbols, $\sigma_{AB}(x) = \sigma_A(x)\sigma_B(x) + \ldots$, and the symbol of the adjoint operator is almost the adjoint of the original symbol, $\sigma_{A^*}(x) = \sigma_A(x)^* + \ldots$. Often just these first term approximations are studied, discarding the remainders. Thus a distorted composition $A \ast B$ can be defined by $\sigma_{A \ast B}(x) := \sigma_B(x)\sigma_A(x)$, and a distorted adjoint $A^*$ by $\sigma_{A^*}(x) := \sigma_A(x)^*$. But it turns out that $\mathcal{L}(\mathcal{H})$ has analogies of all the other Hopf operations as well.
2 Schwartz Kernel Theorem.

Schwartz Kernel Theorem [5]. For nuclear Fréchet spaces and their duals the projective and injective tensor products coincide, and in the sequel \( \hat{\otimes} \) refers to these topological tensor products. Let \( \mathcal{H} \) be a nuclear Fréchet space and \( A \in \mathcal{L}(\mathcal{H}) \). The Schwartz kernel \( K_A \in \mathcal{H} \hat{\otimes} \mathcal{H}' \) of \( A \) is defined by

\[
\langle A\phi, f \rangle =: \langle K_A, f \otimes \phi \rangle
\]

for every \( \phi \in \mathcal{H} \) and \( f \in \mathcal{H}' := \mathcal{L}(\mathcal{H}, \mathbb{C}) \), where the duality brackets are for \( \mathcal{H} \times \mathcal{H}' \) and \( (\mathcal{H} \hat{\otimes} \mathcal{H}') \times (\mathcal{H}' \hat{\otimes} \mathcal{H}) \), respectively. Then the mapping

\[
(A \mapsto K(A) = K_A) : \mathcal{L}(\mathcal{H}) \to \mathcal{H} \hat{\otimes} \mathcal{H}'
\]

is a continuous linear isomorphism, for which

\[
K_{BAC} = (B \otimes C')K_A,
\]

where \( C' \in \mathcal{L}(\mathcal{H}') \) is the adjoint of \( C \), defined by \( \langle \phi, C'f \rangle := \langle C\phi, f \rangle \).

3 Hopf algebras

Basic treatises of Hopf algebras are [2] and [1]. We shall use the following convention: all the vector spaces encountered in this paper are over the complex field \( \mathbb{C} \), and \( V \otimes W \) denotes the tensor product of vector spaces. We shall constantly identify the vector spaces \( \mathbb{C} \otimes V \) and \( V \otimes \mathbb{C} \) with \( V \) by respective mappings \( \mu \otimes v \mapsto \mu v \) and \( v \otimes \mu \mapsto \mu v \). The identity mapping in a vector space is denoted by \( I \). The linear interchanging operator \( \tau : V \otimes W \to W \otimes V \) is defined by \( \tau(v \otimes w) := w \otimes v \).

Algebra. An algebra \( A = (\mathcal{A}, m, \eta) \) consists of a vector space \( \mathcal{A} \) with linear mappings \( m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \) (the product or multiplication) and \( \eta : \mathbb{C} \to \mathcal{A} \) (the unit), satisfying

\[
m(m \otimes I) = m(I \otimes m)
\]

(associativity of the product) and

\[
m(\eta \otimes I) = I = m(I \otimes \eta)
\]

(the unit of the algebra; notice the identifications \( \mathbb{C} \otimes V = V = V \otimes \mathbb{C} \)). We shall use the following abbreviations: \( m(f \otimes g) = fg \) and \( \eta(1) = I \). Then the algebra axioms are written as \( (fg)h = f(gh) \) (\( = fgh \)) and \( I1 = f = fI \). The algebra is commutative if \( m = m\tau \). If \( \mathcal{A}, \mathcal{B} \) are algebras then there is a natural tensor product algebra \( \mathcal{A} \otimes \mathcal{B} \) with the unit

\[
\eta_{\mathcal{A} \otimes \mathcal{B}} := \eta_{\mathcal{A}} \otimes \eta_{\mathcal{B}}
\]

and with the product defined by

\[
m_{\mathcal{A} \otimes \mathcal{B}} := (m_{\mathcal{A}} \otimes m_{\mathcal{B}})(I \otimes \tau \otimes I),
\]

i.e. \( (a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1a_2) \otimes (b_1b_2) \).
Co-algebra. A co-algebra $C = (C, \Delta, \varepsilon)$ consists of a vector space $C$ with linear mappings $\Delta : C \to C \otimes C$ (the co-product or co-multiplication) and $\varepsilon : C \to \mathbb{C}$ (the co-unit) satisfying
\[(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta\]
(co-associativity of the co-product) and
\[(\varepsilon \otimes I)\Delta = I = (I \otimes \varepsilon)\Delta\]
(the co-unit of the co-product). Notice that the co-algebra axioms are obtained by inverting the arrows of the commutative diagrams given by the algebra axioms; this is a dual concept. Co-algebra is called co-commutative if $\Delta = \tau \Delta$. The tensor product co-algebra $C \otimes D$ of co-algebras $C, D$ is endowed with operations
\[\varepsilon_{C \otimes D} := \varepsilon_C \otimes \varepsilon_D\]
and
\[\Delta_{C \otimes D} := (I \otimes \tau \otimes I)(\Delta_C \otimes \Delta_D)\].

Now if $A$ is an algebra and $C$ is a co-algebra, we can define the convolution $A * B$ of operators $A, B \in \mathcal{L}(C, A)$ by $A * B := m(A \otimes B)\Delta$.

Bi-algebra. A bi-algebra $B = (B, m, \eta, \Delta, \varepsilon)$ is an algebra $(B, m, \eta)$ and a co-algebra $(B, \Delta, \varepsilon)$ such that
\[\Delta(fg) = \Delta(f)\Delta(g)\]
(the co-product is multiplicative, or the product is co-multiplicative),
\[\varepsilon(fg) = \varepsilon(f)\varepsilon(g)\]
($\varepsilon$ is a multiplicative linear functional on $B$) and
\[\Delta(I) = I \otimes I\]
(and thereby $\varepsilon(I) = 1$ follows). To state this in another way, $\Delta, \varepsilon$ are algebra morphisms and $m, \eta$ are co-algebra morphisms. Now $\eta \varepsilon \in \mathcal{L}(B)$ is the neutral element with respect to the convolution product, i.e. $A * (\eta \varepsilon) = A = (\eta \varepsilon) * A$, and associativity of the convolution follows directly from both co-associativity of $\Delta$ and associativity of $m$.

Hopf algebra. A Hopf algebra
\[H = (H, m, \eta, \Delta, \varepsilon, S)\]
is a bi-algebra $(H, m, \eta, \Delta, \varepsilon)$ with so called antipode $S \in \mathcal{L}(H)$ such that
\[I * S = \eta \varepsilon = S * I.\]
That is, $S$ is the convolutive inverse of $I$. An involutive Hopf algebra is a Hopf algebra $H$ with involution $j : H \to H$; i.e. $j$ is conjugate-linear, $j^2 = I$, $j(fg) = j(g)j(f)$ and $(j \otimes j)\Delta = \Delta j$. 
**Group algebra example.** Let $G$ be a compact Lie group, $e \in G$ its neutral element. Let $D(G)$ be the space $C^\infty(G)$ with the usual Fréchet space structure. We identify $D(G) \otimes D(G)$ with a subspace of $D(G \times G)$. Then the vector space $D(G)$ is endowed with co-algebra operations $(\Delta \phi)(x, y) := \phi(xy)$ and $\varepsilon(\phi) := \phi(e)$. Hence the co-algebra axioms here correspond to the monoid axioms of the underlying space. Trivially, the usual multiplication and the unit $I \in D(G)$ provide an algebra structure for $D(G)$. Distribution $f \in D'(G)$ of $f, g \in D'(G)$ defined as above coincides with the usual convolution:

$$\langle \phi, f * g \rangle = \int_G \phi(xy) f(x) g(y) \, d\mu_G(x, y)$$

$$= \int_G \phi(x) \int_G f(xy^{-1}) g(y) \, d\mu_G(y) \, d\mu_G(x).$$

We notice that $D(G)$ is in fact a bi-algebra with its canonical mappings, and $\eta(\phi) = \phi(e) I$; the identity element with respect to the convolution is given essentially by the Dirac delta $\delta_e$ at the neutral element $e \in G$. The group bi-algebra $D(G)$ is a natural involutive Hopf algebra with the antipode defined by $(S\phi)(x) := \phi(x^{-1})$ and the involution given by $(j\phi)(x) := \overline{\phi(x)}$. Thus here the antipode axiom is related to the existence and uniqueness of inverse elements in the underlying monoid. Notice also that $D(G) \otimes D(G)$ can be identified with $D(G \times G)$.

**Hopf algebra in a nutshell:** In short, if we denote $m(A \otimes B)\Delta = A * B$ and $I_s = \eta \varepsilon$, the axioms for a Hopf algebra $H$ are:

$$m(m \otimes I) = m(I \otimes m), \quad \eta(1) = I,$$

$$(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta, \quad I * I_s = I = I_s * I,$$

$$\Delta m = m_{\mathcal{H} \otimes \mathcal{H}}(\Delta \otimes \Delta) = (m \otimes m)\Delta_{\mathcal{H} \otimes \mathcal{H}}, \quad \varepsilon m = m_C(\varepsilon \otimes \varepsilon),$$

$$\Delta(I) = I \otimes I, \quad \varepsilon(I) = 1,$$

$$I * S = I_s = S * I,$$

all the mappings $m, \eta, \Delta, \varepsilon, S$ being linear. The axioms for an involution $j$ in an involutive Hopf algebra are then

$$jm = \tau(m \otimes m)j, \quad \Delta j = (j \otimes j)\Delta,$$

$j$ being conjugate-linear.
Consequences of the Hopf axioms. It quite easily follows that $S$ is anti-multiplicative and anti-co-multiplicative,

$$Sm = m(S \otimes S)\tau, \quad \Delta S = \tau(S \otimes S)\Delta,$$

and that

$$SI = I, \quad S\iota = \iota S.$$

Furthermore, in an involutive Hopf algebra, the antipode has the inverse

$$S^{-1} = jSj.$$

Duality of Hopf algebras. Let $H$ be a nuclear Hopf-Fréchet algebra, i.e. a nuclear Fréchet space and a Hopf algebra, with the algebraic tensor products replaced by the topological tensor products in the Hopf definitions. Then the dual space $H' = \mathcal{L}(H, \mathbb{C})$ has a natural dual Hopf algebra structure. Indeed, we define the Hopf structure $(H')' = \mathcal{H}_b \otimes \mathcal{H}_b$ by dualities $H_b \otimes H_b \to \mathbb{C}$ and $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$, where $(\mathcal{H}_b \otimes \mathcal{H}_b)' \cong \mathcal{H}_b \otimes \mathcal{H}_b$ and $\mathbb{C} \cong \mathbb{C}$:

$$\langle \phi, m(f \otimes g) \rangle := \langle \Delta \phi, f \otimes g \rangle,$$

$$\langle \phi, \eta(1) \rangle := \langle \varepsilon(\phi), 1 \rangle = \varepsilon(\phi),$$

$$\langle \phi \otimes \psi, \Delta f \rangle := \langle m(\phi \otimes \psi), f \rangle,$$

$$\langle 1, \varepsilon(f) \rangle := \langle \eta(1), f \rangle,$$

$$\langle \phi, Sf \rangle := \langle S\phi, f \rangle.$$

If $H$ is an involutive Hopf algebra, we can endow the dual with an antipode by

$$\langle \phi, j(f) \rangle := j_{\mathcal{C}}\langle j(S\phi), f \rangle,$$

where $j_{\mathcal{C}} : \mathbb{C} \to \mathbb{C}$ is the complex conjugation $z \mapsto \bar{z}$.

Group algebra dual. Let $G$ be a compact Lie group and $f, g \in \mathcal{D}'(G)$. Then it is easy to verify that $m(f \otimes g) = f * g \in \mathcal{D}'(G)$, and that $\eta(1) = \delta \in \mathcal{D}'(G)$ is the Dirac delta at $e \in G$. Moreover, $\varepsilon f = \int_G f(x) \, d\mu_G(x)$ and $Sf(x) = f(x^{-1})$ informally. The involution for distributions is given by $(j(f))(x) = \overline{f(x^{-1})}$. Notice that $\mathcal{D}'(G) \otimes \mathcal{D}'(G)$ can be identified with $\mathcal{D}'(G \times G)$.

Hopf structures via linear isomorphisms. Let $\mathcal{H}$ be a Hopf algebra, $\mathcal{B}$ a vector space and $\iota : \mathcal{B} \to \mathcal{H}$ a linear bijection. Then this isomorphism naturally endows $\mathcal{B}$ with a Hopf structure:

$$m_{\mathcal{B}} := \iota^{-1}m_{\mathcal{H}}(\iota \otimes \iota),$$

$$\eta_{\mathcal{B}} := \iota^{-1}\eta_{\mathcal{H}},$$

$$\Delta_{\mathcal{B}} := (\iota^{-1} \otimes \iota^{-1})\Delta_{\mathcal{H}}\iota,$$
\[ \varepsilon_B := \varepsilon_{\mathcal{H}t}, \]
\[ S_B := \iota^{-1}S_{\mathcal{H}t}. \]

An involution, if it exists, is defined by
\[ j_B := \iota^{-1}j_{\mathcal{H}t}. \]

In the sequel, we equip \( L(\mathcal{H}) \) with Hopf structures \( L_* \) and \( L_* \) via linear bijections
\[ L_* \overset{\rho}{\rightarrow} L_* \overset{K}{\rightarrow} \mathcal{H} \otimes \mathcal{H}, \]
where \( L_* = L_* = L(\mathcal{H}) \) as topological vector spaces, \( K \) is the Schwartz kernel isomorphism, and \( \rho \) is a natural convolution isomorphism \( A \mapsto A \ast S \).

### 4 Hopf structure via Schwartz kernels

**The fundamental Hopf structure for \( L(\mathcal{H}) \).** Let \( \mathcal{H} \) be a nuclear Hopf-Fr\'echet algebra; the most natural way to endow \( L(\mathcal{H}) \) with a Hopf algebra structure is from \( \mathcal{H} \otimes \mathcal{H}^t \) via the Schwartz kernel isomorphism \( A \mapsto K(A) = K_A \). For instance,
\[ m_*(A \otimes B) := K^{-1}(m(K_A \otimes K_B)). \]

Let us denote this Hopf algebra by \( L_* = (L(\mathcal{H}), m_*, \eta_*, \Delta_*, \varepsilon_*, S_*) \), and write \( \Pi_* := \eta_*(1) \).

**Theorem 1.** The operations in \( L_* \) can be written in terms of the basic Hopf operations \( m, \eta, \Delta, \varepsilon, S \) of \( \mathcal{H} \) as follows:
\[ m_*(A \otimes B) = m(A \otimes B)\Delta = A \ast B, \quad (1) \]
\[ \Pi_* = \eta_*(1) = \eta\varepsilon, \quad (2) \]
\[ \Delta_*(A) = \Delta A, \quad (3) \]
\[ \varepsilon_*(A) = \varepsilon(A(\Pi_\eta)), \quad (4) \]
\[ S_*(A) = SAS. \quad (5) \]

If \( \mathcal{H} \) is an involutive Hopf algebra then \( L_* \) has a Hopf structure with involution \( j_* \), where
\[ j_*(A) = j AjS. \quad (6) \]
Proof. In the following, $A, B \in L(\mathcal{H})$ have respective Schwartz kernels $K_A, K_B \in \mathcal{H} \hat{\otimes} \mathcal{H}'$. Let $f, g \in \mathcal{H}'$ and $\phi, \psi \in \mathcal{H}$. Then

\[
\langle m_*(A \otimes B)\phi, f \rangle := \langle m(K_A \otimes K_B), f \otimes \phi \rangle = \langle K_A \otimes K_B, \Delta(f \otimes \phi) \rangle = \langle (A \otimes B)\Delta\phi, \Delta f \rangle = \langle m(A \otimes B)\Delta\phi, f \rangle = \langle (A \ast B)\phi, f \rangle,
\]

\[
\langle \mathbb{I}_*\phi, f \rangle := \langle \eta(1 \otimes 1), f \otimes \phi \rangle = \langle 1 \otimes 1, \varepsilon(f \otimes \phi) \rangle = \langle \varepsilon\phi, \varepsilon f \rangle = \langle \eta\varepsilon\phi, f \rangle,
\]

\[
\langle \Delta_*(A)(\phi \otimes \psi), f \otimes g \rangle := \langle \Delta(K_A), (f \otimes \phi) \otimes (g \otimes \psi) \rangle = \langle K_A, m((f \otimes \phi) \otimes (g \otimes \psi)) \rangle = \langle Am(\phi \otimes \psi), m(f \otimes g) \rangle = \langle \Delta Am(\phi \otimes \psi), f \otimes g \rangle,
\]

\[
\varepsilon_*(A) := \varepsilon(K_A) = \langle K_A, \eta(1 \otimes 1) \rangle = \langle A\eta 1, \eta 1 \rangle = \langle \varepsilon A\eta 1, 1 \rangle = \varepsilon A\mathbb{I},
\]

\[
\langle S_*(A)\phi, f \rangle := \langle S(K_A), f \otimes \phi \rangle = \langle K_A, S(f \otimes \phi) \rangle = \langle AS\phi, Sf \rangle = \langle SAS\phi, f \rangle.
\]

If $\mathcal{H}$ is an involutive Hopf algebra with involution $j$ then $\mathcal{H}'$ has involution $i$ given by $\langle \phi, i(f) \rangle = \overline{\langle j(S\phi), f \rangle}$; thereby $L_*$ has the Hopf structure with involution $A \mapsto jAjS$, because

\[
\langle (j \otimes i)K_A, f \otimes \phi \rangle = \overline{\langle (S^{-1} \otimes I)K_A, (i \otimes jS)(f \otimes \phi) \rangle} = \overline{\langle S^{-1}Aj S\phi, if \rangle} = \langle jSS^{-1}Aj S\phi, f \rangle = \langle K_{jAjS}, f \otimes \phi \rangle;
\]

notice that we used the fact $K_{BAC} = (B \otimes C')K_A$. $\square$
Hopf operations for $\mathcal{L}(\mathcal{D}(G))$. Let $G$ be a compact Lie group and $A, B \in \mathcal{L}(\mathcal{D}(G))$ with respective Schwartz kernels $K_A, K_B$. Then informally

$$K_{A\ast B}(x, y) = \int G K_A(x, yz^{-1}) K_B(x, z) \, d\mu_G(z),$$

$$\varepsilon_\ast(A) = \int G K_A(e, z) \, d\mu_G(z),$$

$$K_{S_\ast(A)}(x, y) = K_A(x^{-1}, y^{-1}),$$

$$K_{j_\ast(A)}(x, y) = \overline{K_A(x, y^{-1}),}$$

and so on.

### 5 Hopf homomorphism by convolution

In Theorem 1 we equipped $\mathcal{L}(\mathcal{H})$ with the involutive Hopf structure $\mathcal{L}_\ast$ from $\mathcal{H} \hat{\otimes} \mathcal{H}'$ via the Schwartz kernel isomorphism. There the product is the operator convolution $(A, B) \mapsto A \ast B = m(A \otimes B)\Delta$, with the unit element $\mathbb{I}_\ast = \eta \varepsilon \in \mathcal{L}(\mathcal{H})$. We know that $I$ and $S$ are convolution inverses to each other, $I \ast S = \eta \varepsilon = S \ast I$. A simple way to endow $\mathcal{L}(\mathcal{H})$ with an involutive Hopf structure with the unit element $I$ is via the linear bijection

$$\rho = (A \mapsto A \ast S) : \mathcal{L}(\mathcal{H}) \to \mathcal{L}_\ast.$$ 

Alternatively, this Hopf algebra is begotten by the isomorphism

$$L = (A \mapsto L_A = K_{A \ast S}) : \mathcal{L}(\mathcal{H}) \to \mathcal{H} \hat{\otimes} \mathcal{H}'.$$

We denote this Hopf structure by

$$\mathcal{L}_\ast = (\mathcal{L}(\mathcal{H}), m_\ast, \eta_\ast, \Delta_\ast, \varepsilon_\ast, S_\ast, j_\ast).$$

Since $S \ast I = \eta \varepsilon = I$ and $(A \ast S)(\mathbb{I}) = A\mathbb{I}$, we get:

**Theorem 2.** The operations in $\mathcal{L}_\ast$ can be written in terms of the basic Hopf operations $m, \eta, \Delta, \varepsilon, S$ of $\mathcal{H}$ as follows:

$$m_\ast(A \otimes B) = A \ast S \ast B,$$

$$\mathbb{I}_\ast = \eta_\ast(1) = I,$$

$$\Delta_\ast(A) = (\Delta(A \ast S)m) \ast (I \otimes I),$$

$$\varepsilon_\ast(A) = \varepsilon(A(\mathbb{I}_\mathcal{H})),$$

$$S_\ast(A) = (S(A \ast S)S) \ast I.$$

If $\mathcal{H}$ is an involutive Hopf algebra then $\mathcal{L}_\ast$ has a Hopf structure with involution $j_\ast$, where

$$j_\ast(A) = (j(A \ast S)jS) \ast I.$$
6 Pseudodifferential operators

Pseudodifferential operators. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz test function space of rapidly decreasing smooth functions $\mathbb{R}^n \to \mathbb{C}$. An operator $A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n))$ is called a pseudodifferential operator of order $m \in \mathbb{R}$ on $\mathbb{R}^n$, denoted by $A \in \Psi^m(\mathbb{R}^n)$, if it is of the form

$$(Af)(x) = \int_{\mathbb{R}^n} a(x, \xi) \hat{f}(\xi) e^{i2\pi x \xi} \, dx,$$

where the symbol $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies the inequalities

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha \beta} \|\xi\|^{m-|\alpha|}$$

for $\|\xi\|_{\mathbb{R}^n} > 1$, where $C_{\alpha \beta} < \infty$ is a constant depending on $A, m, \alpha, \beta$. Let $M$ be a compact smooth manifold without a boundary, and let $\mathcal{D}(M)$ be the test function space of $C^\infty$-smooth functions on $M$. An operator $A \in \mathcal{L}(\mathcal{D}(M))$ is called a pseudodifferential operator of order $m \in \mathbb{R}$ on $M$, denoted by $A \in \Psi^m(M)$, if all of its localizations belong to $\Psi^m(\mathbb{R}^{\dim(M)})$; this definition makes sense, since $\Psi^m(\mathbb{R}^n)$ is invariant under smooth changes of local coordinates. A principal symbol of a pseudodifferential operator $A \in \Psi^m(M)$ is a function on the cotangent bundle of $M$ defining $A$ up to $\Psi^{m-1}(M)$. If pseudodifferential operators $A \in \Psi^{m_1}(M)$ and $B \in \Psi^{m_2}(M)$ have respective principal symbols $a, b$, then the composition $AB$ has a principal symbol $ab$, and the adjoint $A^*$ has a principal symbol $\overline{a}$; in some sense, pseudodifferential operator algebras behave like function algebras. For more about pseudodifferential calculus, see [4].

Compact Lie groups. The nuclear Fréchet space of interest for us is $\mathcal{D}(G)$, where $G$ is a compact Lie group. Let $\mu_G$ be the normalized Haar measure of $G$. For $A \in \mathcal{L}(\mathcal{D}(G))$, let us define a mapping $s_A : G \to \mathcal{D}'(G)$ and a convolution operator $\sigma_A(x) \in \mathcal{L}(\mathcal{D}(G))$ by

$$(A\phi)(x) = \int_G K_A(x, y) \phi(y) \, d\mu_G(y)$$

$$=: \int_G s_A(x)(xy^{-1}) \phi(y) \, d\mu_G(y)$$

$$= (s_A(x) \ast \phi)(x)$$

$$=: (\sigma_A(x) \phi)(x).$$

Let us call the mapping $\sigma_A : G \to \mathcal{L}(\mathcal{D}(G))$ the symbol of $A$. It is noteworthy that $A \mapsto \sigma_A$ is a one-to-one mapping. For pseudodifferential operators on $G$, there is a symbolic calculus with asymptotic expansions analogous to the Euclidean case, see [3] and [6]. One of the consequences is that if pseudodifferential operators $A_1 \in \Psi^{m_1}(G)$ and $A_2 \in \Psi^{m_2}(G)$ have the respective symbols $\sigma_{A_1}, \sigma_{A_2}$, then the composition $A_1 A_2 \in \Psi^{m_1+m_2}(G)$ has the symbol $x \mapsto \sigma_{A_1}(x) \sigma_{A_2}(x)$ modulo $\Psi^{m_1+m_2-1}(G)$, and the adjoint
\( A_1^* \in \Psi^{m_1}(G) \) has the symbol \( x \mapsto \sigma_{A_1}(x)^* \) modulo \( \Psi^{m_1-1}(G) \). Again, the behavior of pseudodifferential operator algebras resembles function algebra case. Next we show how symbolic calculus is related to the Hopf algebra \( \mathcal{D}(G) \).

**Remark.** The distribution \( L_A \in \mathcal{D}(G) \hat{\otimes} \mathcal{D}'(G) \) introduced in the previous section satisfies \( L_A = (I \otimes S)s_A \), because

\[
L_A(x, y) = K_{A* S}(x, y) = \int_G K_A(x, yz^{-1}) K_S(x, z) \, d\mu_G(z) = K_A(x, yx) = s_A(x, y^{-1}).
\]

Conversely, \( s_A = (I \otimes S)L_A \), since here \( S^2 = I \).

**Theorem 3.** Let \( A, B \in \mathcal{L}(\mathcal{D}(G)) \). Then

\[
\sigma_{A*B}(x) = \sigma_B(x)\sigma_A(x)
\]

for every \( x \in G \). If \( A \in \Psi^{m_1}(G) \) and \( B \in \Psi^{m_2}(G) \) then

\[
A \ast B \in \Psi^{m_1+m_2}(G) \quad \text{and} \quad A \ast B - AB \in \Psi^{m_1+m_2-1}(G).
\]

Moreover, \( \sigma_{I_*}(x) \equiv I \).

**Proof.** Notice that \( \mathcal{D}(G) \) is commutative and that \( S: \mathcal{D}'(G) \to \mathcal{D}'(G) \) is antimultiplicative. Thus we get

\[
s_{A*B} = (I \otimes S)L_{A*B} = (I \otimes S)(L_A L_B) = (I \otimes S)((I \otimes S)s_A)((I \otimes S)s_B)) = ((I \otimes S)(I \otimes S)s_B)((I \otimes S)(I \otimes S)s_A) = s_B s_A,
\]

and consequently \( \sigma_{A*B}(x) = \sigma_B(x)\sigma_A(x) \).

Let \( A \in \Psi^{m_1}(G) \) and \( B \in \Psi^{m_2}(G) \). As it is well-known, \( AB \in \Psi^{m_1+m_2}(G) \) and \( [A, B] = AB - BA \in \Psi^{m_1+m_2-1}(G) \). From the symbolic calculus of [3] and [6] it follows that the operator \( A \ast B \) with the symbol

\[
x \mapsto \sigma_B(x)\sigma_A(x)
\]

belongs to \( \Psi^{m_1+m_2}(G) \), and moreover that \( A \ast B - BA \in \Psi^{m_1+m_2-1}(G) \), because the first term in the asymptotic expansion for \( \sigma_{BA}(x) \) is \( \sigma_B(x)\sigma_A(x) \). Hence also

\[
A \ast B - AB = A \ast B - BA - [A, B]
\]

belongs to \( \Psi^{m_1+m_2-1}(G) \). \( \square \)
Theorem 4. Let $A \in \mathcal{L}(\mathcal{D}(G))$. Then
\[ \sigma_{j_*(A)}(x) = \sigma_A(x)^* \]
for every $x \in G$, where $B^*$ for $B \in \mathcal{L}(\mathcal{D}(G))$ is defined by
\[ \langle \phi, B^* \rangle := \langle B\phi, \cdot \rangle. \]
If $A \in \Psi^m(G)$ then
\[ j_*(A) \in \Psi^m(G) \quad \text{and} \quad j_*(A) - A \in \Psi^{m-1}(G). \]

Proof. Now
\[ s_{j_*(A)} = (I \otimes S)L_{j_*(A)} \]
\[ = (I \otimes S)(j \otimes j)L_A \]
\[ = (I \otimes S)(j \otimes j)(I \otimes S)s_A \]
\[ = (j \otimes SjS)s_A \]
\[ = (j \otimes j)s_A, \]
and combining this with $\langle g*\phi, \cdot \rangle = \langle \phi, f(g)*\cdot \rangle$, we get $\sigma_{j_*(A)}(x) = \sigma_A(x)^*$. If $A \in \Psi^m(G)$ then $A^* \in \Psi^m(G)$ and
\[ x \mapsto \sigma_{A^*}(x) - \sigma_A(x)^* \]
is the symbol of an operator belonging to $\Psi^{m-1}(G)$, by [3] and [6] \(\square\)

Theorem 5. Let $A \in \mathcal{L}(\mathcal{D}(G))$. Then
\[ \sigma_{S_*(A)}(x) = \sigma_A(x^{-1})' \]
for every $x \in G$. If $A \in \Psi^m(G)$ then $S_*(A) \in \Psi^m(G)$.

Proof. Here
\[ s_{S_*(A)} = (I \otimes S)L_{S_*(A)} \]
\[ = (I \otimes S)(S \otimes S)L_A \]
\[ = (I \otimes S)(S \otimes S)(I \otimes S)s_A \]
\[ = (S \otimes S)s_A. \]
Combining this fact with $\langle g\phi, f \rangle = \langle \phi, (Sg)*f \rangle$, we get $\sigma_{S_*(A)}(x) = \sigma_A(x^{-1})'$. Let $A \in \Psi^m(G)$. Then
\[ x \mapsto \sigma_{A^*}(x) - \sigma_A(x)^* \]
is the symbol of an operator belonging to $\Psi^{m-1}(G)$, due to the analogous result for $A \mapsto A^*$ presented in [3] and [6]. If $B \in \Psi^m(G)$ and $\kappa : G \to G$ is $C^\infty$-smooth then
\[ x \mapsto \sigma_B(\kappa(x)) \]
is the symbol of an operator belonging to $\Psi^m(G)$, due to the symbol operator inequalities in [6]. Finally, by choosing $\sigma_B(x) := \sigma_A(x)'$ and $\kappa := (x \mapsto x^{-1})$, we obtain $S_*(A) \in \Psi^m(G)$ \(\square\)
Acknowledgements. The author is grateful to the Academy of Finland for its financial support.

References


A464 Ville Turunen
Sampling at equiangular grids on the 2-sphere and estimates for Sobolev space interpolation
November 2003

A463 Marko Huhtanen, Jan von Pfaler
The real linear eigenvalue problem in $C^n$
November 2003

A462 Ville Turunen
Pseudodifferential calculus on the 2-sphere
October 2003

A461 Tuomas Hytönen
Vector-valued wavelets and the Hardy space $H^1(R^n; X)$
April 2003

A460 Timo Eirola, Jan von Pfaler
Numerical Taylor expansions for invariant manifolds
April 2003

A459 Timo Salin
The quenching problem for the N-dimensional ball
April 2003

A458 Tuomas Hytönen
Translation-invariant Operators on Spaces of Vector-valued Functions
April 2003

A457 Timo Salin
On a Refined Asymptotic Analysis for the Quenching Problem
March 2003

A456 Ville Havu, Harri Hakula, Tomi Tuominen
A benchmark study of elliptic and hyperbolic shells of revolution
January 2003
HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS
RESEARCH REPORTS

The list of reports is continued inside. Electronical versions of the reports are available at http://www.math.hut.fi/reports/.

A470  Lasse Leskelä
       Stabilization of an overloaded queueing network using measurement-based admission control
       March 2004

A469  Jarmo Malinen
       A remark on the Hille–Yoshida generator theorem
       May 2004

A468  Jarmo Malinen, Olavi Nevanlinna, Zhijian Yuan
       On a tauberian condition for bounded linear operators
       May 2004

A467  Jarmo Malinen, Olavi Nevanlinna, Ville Turunen, Zhijian Yuan
       A lower bound for the differences of powers of linear operators
       May 2004

A466  Timo Salin
       Quenching and blowup for reaction diffusion equations
       March 2004

ISBN 951-22-6900-7
ISSN 0784-3143