FAST SOLVERS OF THE
LIPPMANN-SCHWINGER EQUATION

Gennadi Vainikko

Helsinki University of Technology
Institute of Mathematics
Research Reports A387

Espoo, Finland
1997

Abstract. The electromagnetic and acoustic scattering problems for the Helmholtz equation in two and three dimensions are equivalent to the Lippmann-Schwinger equation which is a weakly singular volume integral equation on the support of the scatterer. We propose for the Lippmann-Schwinger equation two discretizations of the optimal accuracy order, accompanied by fast solvers of corresponding systems of linear equations. The first method is of the second order and based on simplest cubatures; the scatterer is allowed to be only piecewise smooth. The second method is of arbitrary order and is based on a fully discrete version of the collocation method with trigonometric test functions; the scatterer is assumed to be smooth on whole space $\mathbb{R}^n$ and of compact support.

AMS subject classifications. 35Q60, 65R20, 65T

Key words. Lippmann-Schwinger equation, Helmholtz equation, cubatures, collocation, fast solvers

ISSN 0784-3143
Libella Painopalvelu Oy, Espoo 1997

Helsinki University of Technology, Institute of Mathematics
P.O.Box 1100, FIN-02015 HUT, Finland
E-mail: Gennadi.Vainikko@hut.fi, math@hut.fi
Fast solvers of the Lippmann–Schwinger equation

G. Vainikko

1 Introduction

In this paper we deal with the integral equation formulation on the scattering problem for the Helmholtz equation in the inhomogeneous media. We assume that the inhomogeneity is smooth or piecewise smooth and of compact support containing the origin, with possibly complex-valued smooth or piecewise smooth refractive index $b : \mathbb{R}^n \to \mathbb{C}$, $n = 2$ or 3, $b(x) = 1$ outside the inhomogeneity. The formulation of the problem reads as follows: find $u : \mathbb{R}^n \to \mathbb{C}$ ($n = 2$ or 3) such that

$$
\Delta u(x) + \kappa^2 b(x) u(x) = 0, \quad x \in \mathbb{R}^n, \quad (1)
$$

$$
u = u^t + u^s, \quad (2)
$$

$$
\lim_{r=|x| \to \infty} r^{(n-1)/2} \left( \frac{\partial u^s}{\partial r} - i\kappa u^s \right) = 0 \quad \text{uniformly for} \quad \frac{x}{|x|} \in S(0, 1) \quad (3)
$$

---

Gennadi Vainikko: Helsinki University of Technology, Institute of Mathematics, P.O.Box 1100, FIN-02015 HUT, Finland; e-mail: Gennadi.Vainikko@hut.fi
where \( u^i \) (the incident field) is a given entire solution of the Helmholtz equation \( \Delta u + \kappa^2 u = 0, \ x \in \mathbb{R}^n \) (usually \( u^i \) is given as a plain wave: \( u^i(x) = \exp(ikd \cdot x) \)), \( d \in \mathbb{R}^n, \ |d| = 1 \), \( u^s \) is the scattered field, \( \kappa > 0 \) is the wave number; (3) means that \( u^s \) must satisfy the Sommerfeld radiation condition. We refer to [2] for more details concerning problem (1)–(3).

Problem (1)–(3) is equivalent to the Lippmann–Schwinger integral equation (see [2])

\[
\begin{align*}
 u(x) &= u^i(x) - \kappa^n \int_{\mathbb{R}^n} \Phi(\kappa|x - y|) a(y)u(y)dy \\
\end{align*}
\]

where \( a = 1 - b \) is smooth or piecewise smooth and of compact support,

\[
\Phi(r) = \begin{cases}
\frac{i}{4}H_0^{(1)}(r), & n = 2 \\
\frac{1}{4\pi}e^{ir}, & n = 3
\end{cases}, \quad r > 0.
\]

\( H_0^{(1)} \) is the Hankel function of the first kind of order zero (see [1], formula 9.1.3). For \( r \to 0, \ H_0^{(1)}(r) \sim -\frac{1}{2\pi} \ln r \). Thus integral equation (4) is weakly singular both in cases \( n = 2 \) and \( n = 3 \). The integration over \( \mathbb{R}^n \) can be replaced by the integration over \( \text{supp} a \).

Problem (1)–(3) and integral equation (4) are uniquely solvable if and only in the homogeneous integral equation corresponding to (4) has only the trivial solution or, equivalently, the homogeneous problem corresponding to, (1)–(3), i.e. the problem with \( u^i = 0 \), has only the trivial solution.

The unboundedness of the domain \( \mathbb{R}^n \) in problem (1)–(3) causes some numerical difficulties. A simplest idea is to use grid methods in a large ball \( B(0, R) \), with boundary condition \( \frac{\partial u}{\partial r} - i\kappa u^s = 0 \) for \( |x| = R \). This method produces very large discrete problems. Another, more popular idea elaborated in [5] is to use coupled finite and boundary methods: in a ball \( B(0, g) \) containing the support of \( a = 1 - b \), the problem is treated by finite elements; a boundary integral equation and the Nyström method are used to treat the problem in the domain \( |x| > g \); finally, a special equation is derived to produce appropriate boundary values of \( u + i\kappa \frac{\partial u}{\partial r} \) for \( |x| = g \). This approach is complicated in it essence. As mentioned in [2], the volume potential approach, i.e. the solution of Lippmann–Schwinger equation (4) instead of (1)–(3), has the advantage that the problem in an unbounded domain is handled in a simple and natural way; a disadvantage is that one has to approximate multidimensional weakly singular integrals and that the discrete problem derived from (4) has a non-sparse matrix. In the present paper we try to show that actually these disadvantages are not serious. First, the optimal convergence order

\[
\|u_N - au\|_\lambda \leq cN^{\lambda-\mu}\|au\|_\mu \quad (0 \leq \lambda \leq \mu)
\]

in the scale of Sobolev norms with any \( \mu > n/2 \) can be achieved by trigonometric collocation method (cf. [6]) applied to a periodized version of (4) if \( a \) and \( u^i \) are sufficiently smooth (\( a \in W^{n,2}(\mathbb{R}^n) \) and \( u^i \in W^{n,2}_{loc}(\mathbb{R}^n) \)). Secondly, the \( N^n \) parameters of \( u_N \) can be computed in \( \mathcal{O}(N^n \ln N) \) arithmetical operations. Finally, the
algorithm needs to store $O(N^n)$ quantities. The method is treated in Section 3. In Section 2 we discuss a method of the second accuracy order:

$$\max_j |u_{j,h} - u(jh)| \leq ch^2 (1 + |\ln h|), \quad h = 1/N;$$

here $a$ may be only piecewise smooth. This method is a modification of a simplest cubature formula method examined in [7] for more general weakly singular integral equations. The purpose of the modification is to obtain a convolution system as the discrete counterpart of (4) maintaining the second order of the approximation. The convolution system can be solved in $O(N^n \log N)$ arithmetical operations using FFT and two grid iterations.

2 The case of piecewise smooth $a$

First we somewhat simplify the form of the integral equation (4). The change of variables

$$\tilde{x} = \kappa x, \quad \tilde{y} = \kappa y, \quad \tilde{u}(\tilde{x}) = u(x), \quad \tilde{a}(\tilde{y}) = a(y), \quad \tilde{u}^i(\tilde{x}) = u^i(x)$$

transforms (4) into

$$\tilde{u}(\tilde{x}) = \tilde{u}^i(\tilde{x}) - \int_{\mathbb{R}^n} \Phi(|\tilde{x} - \tilde{y}|) \tilde{u}(\tilde{y})d\tilde{y}$$

which is (4) with $\kappa = 1$. Thus, without a loss of generality, we put $\kappa = 1$ in (4). To a great wave number $\kappa$ now there corresponds a large support of $\tilde{a}$, namely, $\text{supp} \tilde{a} = \kappa \text{supp} a$. Further, instead of $u^i$, an entire solution to Helmholtz equation, we consider an arbitrary sufficiently smooth function $f : \mathbb{R}^n \to \mathbb{C}$. Thus our problem reads as follows: given a piecewise smooth function $a : \mathbb{R}^n \to \mathbb{C}$ with support in an open bounded set $G \subset \mathbb{R}^n$, and a smooth function $f : \mathbb{R}^n \to \mathbb{C}$, find $u : G \to \mathbb{R}^n$ satisfying the integral equation

$$u(x) = f(x) - \int_{G} \Phi(|x - y|)a(y)u(y)dy \quad (x \in G). \quad (6)$$

Recall that $\Phi$ is given by formula (5). Solving (6) we obtain $u(x)$ for $x \in G$; for $x \in \mathbb{R}^n \setminus G$, $u(x)$ can be obtained after that by simple integration. Notice that due to the Fredholm alternative, equation (6) remains to be uniquely solvable if it is uniquely solvable for $f(x) = u^i(x)$, i.e. if problem (1)–(3) is uniquely solvable. Now we make precise conditions on $a$ used in this section. We assume that

$$a \in W^{2,\infty}(G \setminus \Gamma)$$

where $\Gamma$ consists of a finite number of piecewise smooth compact surfaces $\Gamma_i$ (for $n = 3$) or curves $\Gamma_i$ (for $n = 2$) which may meet each other along manifolds of dimension $\leq n - 2$. More precisely, every $\Gamma_i$ satisfies the following condition (PS): there exist constants $c_0 > 0$ and $r_0 > 0$ such that, for $x \in \Gamma_i$, the piece $\Gamma_i \cap B(x, r_0)$ of
\( \Gamma_i \) is representable in the form \( z_n = \varphi(z') \), \( z' = (z_1, \ldots, z_{n-1}) \in Z_{i,x} \) where \( z_1, \ldots, z_n \) is a suitable orthogonal system of coordinates obtained from the original system \( x_1, \ldots, x_n \) by the translation of the origin into the point \( x \) and a rotation of axes; \( Z_{i,x} \subset \mathbb{R}^{n-1} \) is a bounded closed region and \( \varphi = \varphi_{i,x} \) is a continuous function on \( Z_{i,x} \) which is continuously differentiable with \( |\text{grad}\varphi(z')| \leq c_0 \) everywhere in \( Z_{i,x} \) except possibly a manifold of dimension \( \leq n - 2 \). The surfaces of a ball, cylinder, cone and cube are simplest examples of \( \Gamma_i \) satisfying \( (PS) \), together they may build more complicated configurations, e.g. two tangential balls (one in another).

Let \( h > 0 \) be a discretization step. For \( j = (j_1, \ldots, j_n) \in \mathbb{Z}^n \), denote

\[
B_{j,h} = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \left( j_k - \frac{1}{2} \right) h < x_k < \left( j_k + \frac{1}{2} \right) h, \ k = 1, \ldots, n \right\}.
\]

This is a rectangular cell with the center at \( jh \). We define the grid approximation of \( a \) as follows. We put

\[
a_{j,h} = a(jh) \quad \text{if} \quad B_{j,h} \cap \Gamma = \emptyset;
\]

in particular,

\[
a_{j,h} = 0 \quad \text{if} \quad B_{j,h} \cap \text{supp} \ a = \emptyset.
\]

For \( B_{j,h} \cap \Gamma \neq \emptyset \) we put

\[
a_{j,h} = h^{-n} \sum_{p=1}^{q_j} a(x_{j,h}^{(p)}) \text{ meas } B_{j,h}^{(p)} \quad \text{with some } x_{j,h}^{(p)} \in B_{j,h}^{(p)}
\]

where \( B_{j,h}^{(p)} \) (\( p = 1, \ldots, q_j \)) are the connectivity components of the open set \( B_{j,h} \setminus \Gamma \).

The measure of \( B_{j,h}^{(p)} \) may be computed approximately with an accuracy \( O(h^{n+1}) \).

Further, define

\[
\Phi_{j,h} = \begin{cases} 
\Phi(j|h), & 0 \neq j \in \mathbb{Z}^n, \\
0, & j = 0.
\end{cases}
\]

Take a sufficiently large \( N \in \mathbb{N} \) and an open bounded set \( G \in \mathbb{R}^n \) (independent of \( h \)) such that

\[
\text{supp} \ a \subset \sum_{j \in Z_N^n} \overline{B}_{j,h} \subset G \quad \text{(7)}
\]

where

\[
Z_N^n = \left\{ j \in \mathbb{Z}^n : -\frac{N}{2} < j_k \leq \frac{N}{2}, \ k = 1, \ldots, n \right\}.
\]

We approximate (6) by the discrete problem

\[
u_{j,h} = f(jh) - h^n \sum_{k \in Z_N^n} \Phi_{j-k,h} a_{k,h} u_{k,h} \quad (j \in Z_N^n).
\]

This is a modification of the cubature formula method examined in [7]. The modification is performed so that the discrete problem maintains the convolution structure.

The following convergence result can be followed from the general result of [7].
Theorem 1 Assume that \( \text{supp} \ a \) satisfies (7), \( f \in C^2(G) \) and \( a \in W^{2,\infty}(G \setminus \Gamma) \) with \( \Gamma \) satisfying condition (PS). Finally, assume that the homogeneous integral equation corresponding to (6) possesses only the trivial solution. Then system (8) is uniquely solvable for all sufficiently small \( h > 0 \), and
\[
\max_{j \in \mathbb{Z}_h^n} |u_{j,h} - u(jh)| \leq ch^2(1 + |\ln h|)
\]
where \( u \in C(G) \) is the solution of (6) and \( u_{j,h} (j \in \mathbb{Z}_h^n) \) is the solution of system (8).

An application of the convolution multi-matrix of system (8) to the multi-vector \( a_h u_h \) costs \( O(N^n \log N) \) arithmetical operations if FFT techniques is involved. This enables to solve system (8) with the accuracy \( O(h^2(1 + |\ln h|)) \) in \( O(N^n \ln N) \) arithmetical operations using two-grid iterations. For instance, putting \( M \sim N^{1/3} \) for the coarse grid, 5 iterations are sufficient; the \( M \)-multisystems can be solved e.g. by Gauss elimination. There are different other strategies for the two-grid methods. We quote to [7] for numerical schemes. In Section 3.7 we present more details in the case of smooth \( a \) and trigonometric collocation as the basis of the discretization.

3 The case of a smooth \( a \)

3.1 Periodization of the problem

From now we assume that
\[
\text{supp} \ a \subset \overline{B}(0, \rho), \quad a \in W^{n,2}(\mathbb{R}^n), \quad f \in W^{\mu,2}_{\text{loc}}(\mathbb{R}^n), \quad \mu > \frac{n}{2}.
\]
Due to the Sobolev imbedding theorem, it follows from (9) that \( a \) and \( f \) are continuous on \( \mathbb{R}^n \). Denote
\[
G_R = \{ x \in \mathbb{R}^n : |x_k| < R, \ k = 1, \ldots, n \}
\]
where \( R \geq 2\rho \) is a parameter. Multiplying both sides of (6) by \( a(x) \) we rewrite equation (6) with respect to \( a(x)u(x) \):
\[
a(x)u(x) = a(x)f(x) - a(x) \int_{G_R} \Phi(|x-y|)[a(y)u(y)]dy \quad (x \in G_R).
\]
We are interested in finding of \( a(x)u(x) \) for \( x \in \text{supp} \ a \subset \overline{B}(0, \rho) \). For those \( x \), only values from \( \overline{B}(0, 2\rho) \) of the function \( \Phi(|x|) \) are involved; changing \( \Phi \) outside this ball, the solution \( a(x)u(x) \) does not change in \( \overline{B}(0, \rho) \). We exploit this observation and define a new kernel \( K(x) \) which coincides with \( \Phi(|x|) \) for \( x \in \overline{B}(0, 2\rho) \). The simplest possibility is to cut \( \Phi(|x|) \) off at \( |x| = R \):
\[
K(x) = \begin{cases} 
\Phi(|x|), & |x| \leq R \\
0, & x \in G_R \setminus B(0, R), \quad R \geq 2\rho.
\end{cases}
\]
We also consider the possibility with smooth cutting:

\[ K(x) = \Phi(|x|)\psi(|x|), \quad x \in G_R, \quad R > 2\varrho, \]  

(11) 

with \( \psi : [0, \infty) \to \mathbb{R} \) satisfying the conditions

\[ \psi \in C^\infty[0, \infty), \quad \psi(r) = 1 \text{ for } 0 \leq r \leq 2\varrho, \quad \psi(r) = 0 \text{ for } r \geq R. \]  

(12) 

After that we extend functions \( K, a \) and \( af \) from \( G_R \) to \( \mathbb{R}^n \) as \( 2R \)-periodic functions with respect to \( x_1, \ldots, x_n \); for extensions we use the same designations. Thus we have a multiperiodic integral equation

\[ v(x) = a(x)f(x) - a(x) \int_{G_R} K(x - y)v(y)dy. \]  

(13) 

It is easy to see that a unique solvability of (6) involves a unique solvability of (13). As already explained, the solutions are related by

\[ v(x) = a(x)u(x) \text{ for } x \in \overline{B}(0, \varrho), \]

moreover, \( v \) is the \( 2R \)-periodization of \( au \) restricted to \( G_R \). Further,

\[ u(x) = f(x) - \int_{B(0, \varrho)} \Phi(|x - y|)v(y)dy \text{ for } x \in \mathbb{R}^n \]  

(14) 

and in particular

\[ u(x) = f(x) - \int_{G_R} K(x - y)v(y)dy \text{ for } x \in \overline{B}(0, \varrho). \]

3.2 Periodic Sobolev spaces \( H^\lambda \)

The trigonometric orthonormal basis of \( L^2(G_R) \) is given by

\[ \varphi_j(x) = (2R)^{-n/2} \exp(i\pi j \cdot \frac{x}{R}), \quad j = (j_1, \ldots, j_n) \in \mathbb{Z}^n. \]  

(15) 

Introduce the Sobolev space \( H^\lambda = H^\lambda(G_R) \), \( \lambda \in \mathbb{R} \), which consists of \( 2R \)-multiperiodic functions (distributions) \( u \) having the finite norm

\[ \|u\|_\lambda = \left( \sum_{j \in \mathbb{Z}^n} 2^{2\lambda} |\hat{u}(j)|^2 \right)^{1/2}. \]

Here

\[ \hat{u}(j) = \int_{G_R} u(x)\varphi_j(x)dx = \langle u, \varphi_{-j} \rangle, \quad j \in \mathbb{Z}^n, \]

are the Fourier coefficients of \( u \), and

\[ j = \begin{cases} |j| = (j_1^2 + \ldots + j_n^2)^{1/2}, & 0 \neq j \in \mathbb{Z}^n \\ 1, & j = 0. \end{cases} \]
Notice that $H^\lambda \subset W^{\lambda,2}_{\text{loc}}(\mathbb{R}^n)$, thus a function $u \in H^\lambda$ with $\lambda > n/2$ is continuous. We will also use the relation

$$u, v \in H^\lambda, \quad \lambda > n/2 \Rightarrow uv \in H^\lambda, \quad \|uv\|_\lambda \leq c_\lambda \|u\|_\lambda \|v\|_\lambda;$$

a proof can be constructed as in [4] where somewhat different Sobolev spaces are used.

### 3.3 Trigonometric collocation

Recall the designation

$$Z_N^n = \{ j \in \mathbb{Z}^n : -\frac{N}{2} < j_k \leq \frac{N}{2}, \quad k = 1, \ldots, n \}.$$ 

Let $\mathcal{T}_N$ be the $N^n$-dimensional linear space of trigonometric polynomials of the form $v_N = \sum_{j \in Z_N^n} c_j \varphi_j$, $c_j \in \mathbb{C}$. The formula

$$P_N v = \sum_{j \in Z_N^n} \hat{v}(j) \varphi_j$$

defines the orthogonal projection $P_N$ in $H^\lambda$ to $\mathcal{T}_N$. Clearly,

$$\|v - P_N v\|_\lambda \leq \left(\frac{N}{2}\right)^{\lambda-\mu} \|v\|_\mu \quad \text{for} \quad \lambda \leq \mu, \quad \lambda, \mu \in \mathbb{R}. \quad (16)$$

For $v \in H^n, \mu > \frac{n}{2}$, we define the interpolation projection $Q_N v$ claiming

$$Q_N v \in \mathcal{T}_N, \quad (Q_N v)(jh) = v(jh), \quad j \in Z_N^n, \quad \text{where} \quad h = 2R/N.$$

The error of the trigonometric interpolation can be estimated by (cf. (16))

$$\|v - Q_N v\|_\lambda \leq c_{\lambda,\mu} N^{\lambda-\mu} \|v\|_\mu \quad \text{for} \quad 0 \leq \lambda \leq \mu, \quad \mu > \frac{n}{2}; \quad (17)$$

a proof with a characterization of the constant $c_{\lambda,\mu}$ can be constructed following [8].

We solve the equation (13) by trigonometric collocation method

$$v_N = Q_N (af) - Q_N (aKv_N) \quad (18)$$

where $K$ is the integral operator from (13):

$$(Kv)(x) = \int_{G_R} K(x - y)v(y)dy.$$

Since $K(x)$ is $2R$-periodic, the eigenvalues and eigenfunctions of the convolution operator $K$ are known to be $\hat{K}(j)$ and $\varphi_j(x)$, respectively:

$$K\varphi_j = \hat{K}(j)\varphi_j \quad (j \in \mathbb{Z}^n).$$
In the case of cutting (10), closed formulae for $\hat{K}(j)$ are presented in Section 3.8. According to those formulae,

$$|\hat{K}(j)| \leq c_R \begin{cases} |j|^{-3/2}, & n = 2 \\ |j|^{-2}, & n = 3 \end{cases} \quad (j \neq 0).$$

In the case of the smooth cutting (11),

$$|\hat{K}(j)| \leq c_R |j|^{-2} \quad (j \neq 0) \text{ for } n = 2 \text{ and } n = 3.$$  

We present in Section 3.8 a cheap algorithm to approximate $\hat{K}(j)$, $j \in \mathbb{Z}_N^n$, in this case.

We quote to [6] for the study of the trigonometric collocation method in a more complicated but one dimensional situation.

### 3.4 Matrix form of the collocation method

We have two representations of a trigonometric polynomial $v_N \in \mathcal{T}_N$: (i) through its Fourier coefficients by

$$v_N(x) = \sum_{k \in \mathbb{Z}_N^n} \hat{v}_N(k) \varphi_k(x)$$

with $\varphi_k$ defined in (15); (ii) through its nodal values by

$$v_N(x) = \sum_{j \in \mathbb{Z}_N^n} v_N(jh) \varphi_{N,j}(x), \quad h = \frac{2R}{N},$$

with $\varphi_{N,j} \in \mathcal{T}_N$ satisfying $\varphi_{N,j}(kh) = \delta_{j,k}$ (the Kronecker symbol), $j, k \in \mathbb{Z}_N^n$. An explicit formula for $\varphi_{N,j}$ is given by

$$\varphi_{N,j}(x) = N^{-n} \sum_{k \in \mathbb{Z}_N^n} \exp \left( i\pi k \cdot \left( \frac{x}{R} - \frac{2j}{N} \right) \right), \quad j \in \mathbb{Z}_N^n.$$ 

It is easy to change the type of representation. Having the nodal values $\underline{v}_N$ of $v_N \in \mathcal{T}_N$, its Fourier coefficients $\hat{v}_N = \mathcal{F}_N \underline{v}_N$ are given by the discrete Fourier transformation $\mathcal{F}_N$:

$$\hat{v}_N(k) = \int_{G_R} v_N(x) \varphi_{-k}(x) dx = (2R)^{n/2} N^{-n} \sum_{j \in \mathbb{Z}_N^n} v_N(jh) \exp \left( -i\pi k \cdot \frac{2j}{N} \right), \quad k \in \mathbb{Z}_N^n.$$ 

Conversely, if we have the Fourier coefficients $\hat{v}_N$ of $v_N \in \mathcal{T}_N$, then its nodal values $\underline{v}_N = \mathcal{F}_N^{-1} \hat{v}_N$ are given by the inverse discrete Fourier transformation $\mathcal{F}_N^{-1}$:

$$v_N(jh) = (2R)^{-n/2} \sum_{k \in \mathbb{Z}_N^n} \hat{v}_N(k) \exp \left( i\pi k \cdot \frac{2j}{N} \right), \quad j \in \mathbb{Z}_N^n.$$
Using the transforms $F_N$ and $F_N^{-1}$, the collocation method (18) can be represented as

$$v_N = (af)_N - a_N F_N^{-1} K_N F_N y_N,$$

where $a_N$ and $(af)_N$ present the nodal values at $jh$, $j \in \mathbb{Z}_N^*$, of $a$ and $af$, respectively, and $K_N$ multiplies the Fourier coefficients $\hat{v}_N(k)$ of $\hat{v}_N = F_N y_N$ by $\hat{K}(k)$, $k \in \mathbb{Z}_N$; the product of $a_N$ and $F_N^{-1} K y_N$ is to be taken pointwise at nodes. Thus, the matrix form of the collocation method (18) is given by

$$A_N y_N = g_N, \quad A_N = I_N + a_N F_N^{-1} \hat{K}_N F_N, \quad g_N = (af)_N.$$

(19)

An application of $F_N$ or $F_N^{-1}$ to a $n^n$-vector $y_N$ or $\hat{v}_N$ in a usual way costs $N^{2n}$ multiplications and additions but the FFT does this in $O(n \log N)$ arithmetical operations. The application of diagonal operations $K_N$ and $a_N$ cost $N^n$ multiplications. Thus, the application of $A_N$ is cheap and therefore it is appropriate to solve (19) by some iteration method. We return to this question in Sections 3.6–3.7.

With respect to Fourier coefficients, the matrix form of the collocation method (18) reads as follows:

$$\hat{A}_N \hat{v}_N = \hat{g}_N, \quad \hat{A}_N = I_N + F_N a_N F_N^{-1} \hat{K}_N, \quad \hat{g}_N = F_N (af)_N.$$

3.5 Convergence of the collocation method

Lemma 1 Assume that $a \in H^\mu$, $\mu > n/2$, and $|\hat{K}(j)| \leq c|j|^{-2}$ ($j \neq 0$). Then

$$\|aK - Q_N aK\|_{L(H^\lambda, H^\lambda)} \leq c \begin{cases} N^{\lambda - \mu}, & 0 \leq \lambda \leq \mu, \mu \leq 2, \\ N^{-2}, & 0 \leq \lambda \leq \mu - 2, \mu \geq 2, \\ N^{\lambda - \mu}, & \mu - 2 \leq \lambda \leq \mu, \mu \geq 2. \end{cases}$$

(20)

Proof. First notice that $K \in L(H^\lambda, H^{\lambda+2})$ for any $\lambda \in \mathbb{R}$. Consider the case $\mu \geq 2$, $0 \leq \lambda \leq \mu - 2$. Due to (17),

$$\|aKv - Q_N(aKv)\|_\lambda \leq cN^{-2}\|aKv\|_{\lambda+2} \leq c'N^{-2}\|a\|_{\lambda+2}\|Kv\|_{\lambda+2} \leq c''N^{-2}\|a\|_{\lambda+2}\|v\|_{\lambda}$$

resulting to $\|aK - Q_N aK\|_{L(H^\lambda, H^\lambda)} \leq cN^{-2}$. The other cases can be analysed in a similar way. □

In the case of cutting (10), $n = 2$, we have $|\hat{K}(j)| \leq c|j|^{-3/2}$ ($j \neq 0$), and instead of (20) we obtain

$$\|aK - Q_N aK\|_{L(H^\lambda, H^\lambda)} \leq c \begin{cases} N^{\lambda - \mu}, & 0 \leq \lambda \leq \mu, \mu \leq \frac{3}{2}, \\ N^{-3/2}, & 0 \leq \lambda \leq \mu - \frac{3}{2}, \mu \geq \frac{3}{2}, \\ N^{\lambda - \mu}, & \mu - \frac{3}{2} \leq \lambda \leq \mu, \mu \geq \frac{3}{2}. \end{cases}$$

(21)

Theorem 2 Assume that the functions $a$ and $f$ satisfy (9), and the homogeneous problem corresponding to (1)-(3), with $\kappa = 1$, $w = 0$, has only the trivial solution. Then equation (13) has a unique solution $v \in H^\mu$, collocation equation (18) has a unique solution $v_N \in \mathcal{T}_N$ for $N \geq N_0$, and

$$\|v_N - v\|_\lambda \leq c\|v - Q_N v\|_\lambda \leq c'\|v\|_\mu N^{\lambda - \mu}, \quad 0 \leq \lambda \leq \mu.$$

(22)
\textbf{Proof.} The bounded inverse to $I + aK$ in $\mathcal{L}(H^\lambda, H^\lambda)$ exists since $aK \in \mathcal{L}(H^\lambda, H^\lambda)$ is compact and the homogeneous integral equation corresponding to (13) has only the trivial solution. Using (20) or (21) we obtain that the inverse to $I + Q_N aK$ in $\mathcal{L}(H^\lambda, H^\lambda)$ exists for all sufficiently great $N$, and

$$\|(I + Q_N aK)^{-1}\|_{\mathcal{L}(H^\lambda, H^\lambda)} \leq c \quad (0 \leq \lambda \leq \mu, \quad N \geq N_0).$$

(23)

Error estimate (22) follows form (23) and the equality

$$(I + Q_N aK)(v_N - v) = Q_N (af) - v - Q_N aK v = Q_N v - v. \quad \square$$

Thus we have an approximation $v_N \in \mathcal{T}_N$ to the $2R$-periodic extension $v$ of $au$. An approximation to $u$ outside $B(0, \varrho)$ can be defined by the discretization of (14):

$$u_N(x) = f(x) - h^n \sum_{j \in \mathbb{Z}_N^n} \Phi(|x - jh|)v_N(jh), \quad h = 2R/N, \quad |x| > \varrho.$$

It can be deduced from (22) that

$$|u_N(x) - u(x)| \leq c|x|^{-(n-1)/2}\|v\|_{\mu} N^{-\mu}, \quad |x| \geq 2\varrho,$$

(24)

where the constant $c$ is independent of $x$ and $N$ (this constant has a bad behaviour as $|x| \to \varrho$, therefore we restricted it to $|x| \geq 2\varrho$).

The following well known asymptotic formula for the solution of (6) follows from (14) and the properties of $\Phi(r)$ as $r \to \infty$ (for the behaviour of $H^1_0(r)$, see formula 9.2.3 in [1]):

$$u(x) = f(x) - \frac{e^{i|x|}}{|x|^{(n-1)/2}} u_\infty(\hat{x}) + O(|x|^{-(n+1)/2}), \quad |x| \to \infty,$$

where $\hat{x} = x/|x|$ and the far field pattern $u_\infty$ is defined by

$$u_\infty(\hat{x}) = \gamma_n \int_{G_R} e^{-i\hat{x} \cdot y} v(y) dy,$$

$$\gamma_n = \begin{cases} 1 + i/4\sqrt{\pi}, & n = 2, \\ 1/4\pi, & n = 3. \end{cases}$$

A natural approximation to $u_\infty$ is given by

$$u_{\infty,N}(\hat{x}) = \gamma_n h^n \sum_{j \in \mathbb{Z}_N^n} e^{-i\hat{x} \cdot jh} v_N(jh), \quad \hat{x} \in S(0,1).$$

Under conditions of Theorem 2,

$$\max_{\hat{x} \in S(0,1)} |u_{\infty,N}(\hat{x}) - u_\infty(\hat{x})| \leq c\|v\|_\mu N^{-\mu}.$$

(25)
3.6 Solution of the system of the collocation method

As already mentioned, iteration methods are most natural to solve the system (19). Due to (23), for the condition number \( \gamma_N \) of the system, with respect to the spectral norm, we have

\[
\gamma_N := \| A_N \| \| A_N^{-1} \| = \| I + Q_N a K \|_{\mathcal{L}(H^0, H^0)} \| (I + Q_N a K)^{-1} \|_{\mathcal{L}(H^0, H^0)} \leq \gamma
\]

where the constant \( \gamma \) is independent of \( N \). If \( v^k_N \) denotes the \( k \)th iteration approximation by the conjugate gradient method applied to the symmetrized system

\[
A^*_N A_N v_N = A^*_N g_N, \quad A^*_N = I_N + F^{-1}_N \hat{K}^*_N F_N u^*_n,
\]

then (see e.g. [3])

\[
\| v^k_N - v_N \|_0 \leq c q^k \| v^0_N - v_N \|_0, \quad q = (\gamma - 1)/(\gamma + 1),
\]

and the accuracy \( \| v^k_N - v_N \|_0 \leq c N^{-\mu} \) (cf. (22)) will be achieved in \( \mathcal{O}(\log N/\log q) \) iteration steps. Since every iteration step costs \( \mathcal{O}(N^n \log N) \) arithmetical operations, the whole cost of the method is \( \mathcal{O}(N^n \log^2 N) \) arithmetical operations. This amount of the work can be reduced to \( \mathcal{O}(N^n \log N) \) arithmetical operations with the help of two grid iteration schemes.

3.7 Two grid iterations

Denoting

\[
g_N = Q_N(a f) \in T_N, \quad T_N = Q_N a K \in \mathcal{L}(H^\lambda, H^\lambda),
\]

the collocation equation (18) can be rewritten as \( v_N + T_N v_N = g_N \). Take a \( M \in \mathbb{N} \) of order \( M \sim N^\Theta, \ 0 < \Theta < 1 \). The collocation equation is equivalent to \( (I + T_M)^{-1}(I + T_N)v_N = (I + T_M)^{-1}g_N \), or

\[
v_N = T_{M,N}v_N + g_{M,N}
\]

with

\[
T_{M,N} = (I + T_M)^{-1}(T_M - T_N), \quad g_{M,N} = (I + T_M)^{-1}g_N.
\]

Under conditions of Lemma 1 and Theorem 2 we have for \( 0 \leq \lambda \leq \mu - 2, \mu \geq 2 \), the estimate (see (20) and (23))

\[
\| T_{M,N} \|_{\mathcal{L}(H^\lambda, H^\lambda)} \leq c M^{-2} \leq c N^{-2\Theta}.
\]

Thus, the norm of the operator \( T_{M,N} \) is small, and we may apply the iterations

\[
v^k_N = T_{M,N}v^{k-1}_N + g_{M,N} \quad (k = 1, 2, \ldots)
\]

starting e.g. from \( v^0_N = v_M = (I + T_M)^{-1}g_M \). For the exact collocation solution \( v_N \) we have \( v_N = T_{M,N}v_N + g_{M,N} \) and

\[
v^k_N - v_N = T_{M,N}(v^{k-1}_N - v_N) = \ldots = T_{M,N}^k(v^0_N - v_N),
\]

\[
\| v^k_N - v_N \|_\lambda \leq \| T_{M,N} \|_{\mathcal{L}(H^\lambda, H^\lambda)}^k(\| v_N - v \|_\lambda + \| v_M - v \|_\lambda)
\]

\[
\leq c c^k N^{-2\Theta k + \Theta(\lambda - \mu)}\| v \|_\mu \leq \varepsilon N^{\lambda - \mu} \| v \|_\mu \quad (0 \leq \lambda \leq \mu)
\]
with a small $\varepsilon > 0$ (cf. (22)) provided that $(2k + \mu - \lambda)\Theta > \mu - \lambda$. This condition is most strong for $\lambda = 0$:

$$k > \frac{1 - \Theta}{2\Theta} \mu \quad \text{for fixed } \Theta \in (0, 1),$$

or equivalently

$$\Theta > \frac{\mu}{\mu + 2k} \quad \text{for fixed } k \in \mathbb{N}.$$  

So only few iterations (26) are needed to achieve the accuracy (22) by $v^k_N$, and this number of iterations may be taken to be independent of $N$. On the other hand, if we put $\Theta > \frac{\mu}{\mu + 2}$, then only one iteration (26) is sufficient, i.e. asymptotically already $v^1_N$ achieves the accuracy (22).

To present the matrix form of the two-grid iterations (26), notice that

$$(I + T_M)^{-1} = I - (I + T_M)^{-1}T_M.$$  

Thus (26) can be written in the form where $(I - T_M)^{-1}$ is applied only to functions from $T_M$:

$$v^k_N = [I - (I + T_M)^{-1}T_M][(T_M - T_N)v^{k-1}_N + g_N].$$

With respect to the Fourier coefficients of $v^k_N$, the matrix form of the two-grid iterations (26) is as follows:

$$\hat{v}^k_N = [I_N - \hat{P}_{N,M}\hat{A}_M^{-1}\mathcal{F}_M\mathcal{G}_M R_{M,N}\mathcal{F}_N^{-1}\hat{K}_N][\hat{g}_N + (\hat{P}_{N,M}\mathcal{F}_M\mathcal{G}_M R_{M,N} - \mathcal{F}_N\mathcal{G}_N)\mathcal{F}_N^{-1}\hat{K}_N\hat{v}^{k-1}_N]. \quad (27)$$

The designations $\hat{A}_N$, $\mathcal{F}_N$, $\mathcal{F}_N^{-1}$, $\hat{K}_N$, $\hat{g}_N$, $g_N$ have been explained in Section 3.4; $R_{M,N}\mathcal{G}_N$ restricts $\mathcal{G}_N$ from the net $hZ^n_N = \{hj : j \in Z^n_M\}$, $h = \frac{2R}{N}$, to the subnet $h'Z^n_M$, $h' = \frac{2R}{M}$ (we assume that $h'/h = N/M$ is an integer); the prolongation operator $\hat{P}_{N,M}$ is defined by

$$(\hat{P}_{N,M}\hat{w}_M)(j) = \begin{cases} \hat{w}_M(j), & j \in Z^n_M, \\ 0, & j \in Z^n_M \setminus Z^n_M. \end{cases}$$

Of course, a vector $u_M = \hat{A}_M^{-1}w_M$ is computed solving the $M^n$-system $\hat{A}_M u_M = w_M$. This can be done e.g. by the conjugate gradient method in $\mathcal{O}(M^n \log^2 M) = \mathcal{O}(N^\Theta n \log^2 N)$ arithmetical operations as explained in Section 3.6. For $0 < \Theta \leq 1/3$, also a direct solution of the $M^n$-system e.g. by the Gauss elimination holds the amount of work in $\mathcal{O}(N^n)$ arithmetical operations.

Most costful operations in (27) are $\mathcal{F}_N$ and $\mathcal{F}_N^{-1}$. During one iterations, they occur three times, plus once to compute $\hat{g}_N$. Asymptotically most cheap version of (27) is obtained putting $\Theta > \mu/(\mu + 2)$. As explained, then only one iteration (26) is sufficient to achieve the accuracy (22); respectively, only once we have to solve $M^n$-system in iteration(27), and once it should be done to compute the initial guess $\hat{v}^0_N = \hat{v}_M = \hat{A}_M^{-1}\hat{g}_M$. The whole amount of the computational work is $\mathcal{O}(N^n \log N)$ arithmetical operations, and it is caused by 4 operations with $\mathcal{F}_N$ and $\mathcal{F}_N^{-1}$; all other operations cost $\mathcal{O}(N^n)$ or less.

Recall that this analysis is based on (20) for $0 \leq \lambda \leq \mu - 2$, $\mu \geq 2$. It is easy to complete the analysis considering other cases in (20) and (21).
3.8 Appendix: Fourier coefficients of $K(x)$

Clearly, $(\Delta + 1) \varphi_j = (1 - \pi^2 |j|^2 / R^2) \varphi_j$. For $\pi |j| \neq R$, denoting $\lambda_j = R^2 / (\pi^2 |j|^2 - R^2)$, with help of the Green formula we obtain

$$
\dot{K}(j) = \int_{G_R} K(x) \varphi_{-j}(x) dx = -\lambda_j \int_{G_R} K(x)(\Delta + 1) \varphi_{-j}(x) dx
$$

$$
= -\lambda_j \lim_{\delta \to 0} \int_{B(0,R) \setminus B(0,\delta)} K(x)(\Delta + 1) \varphi_{-1}(x) dx
$$

$$
= -\lambda \lim_{\delta \to 0} \left\{ \left( \int_{S(0,R)} - \int_{S(0,\delta)} \right) \left( K \frac{\partial \varphi_{-j}}{\partial r} - \frac{\partial K}{\partial r} \varphi_{-j} \right) dS + \int_{B(0,R) \setminus B(0,\delta)} ((\Delta + 1)K(x)) \varphi_{-j}(x) dx \right\}
$$

where $\frac{\partial}{\partial r} = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}$. According to the construction (see (10) and (11)) $K(x) = \Phi(|x|)$ on the sphere ($n = 3$) or circle ($n = 2$) $S(0, \delta)$. Taking into account the asymptotics of $\Phi(r)$ and $\Phi'(r)$ as $r \to 0$, we obtain

$$
\dot{K}(j) = \lambda_j \left\{ \varphi_{-j}(0) - \int_{S(0,R)} \left( K \frac{\partial \varphi_{-j}}{\partial r} - \frac{\partial K}{\partial r} \varphi_{-j} \right) dS - \lim_{\delta \to 0} \int_{B(0,R) \setminus B(0,\delta)} ((\Delta + 1)K(x)) \varphi_{-j}(x) dx \right\}. \tag{28}
$$

Cutting (10), $n = 3$. According to (10), since $(\Delta + 1)K(x) = (\Delta + 1)\Phi(|x|) = 0$ for $0 \neq x \in B(0,R)$, (28) reduces to

$$
\dot{K}(j) = \lambda \left\{ \varphi_{-j}(0) - \Phi(R) \int_{S(0,R)} \frac{\partial \varphi_{-j}}{\partial r} dS + \Phi'(R) \int_{S(0,R)} \varphi_{-j} dS \right\}.
$$

We use the symmetry argument to evaluate

$$
\int_{S(0,R)} e^{i\pi j \cdot x / R} dS = R^2 \int_{S(0,1)} e^{i\pi j \cdot x} dS = R^2 \int_{S(0,1)} e^{i\pi |x_1|} dS
$$

$$
= R^2 \int_{-1}^{1} e^{i\pi |x_1|} 2\pi dx_1 = \frac{4R^2}{|j|} \sin(\pi |j|), \quad j \neq 0.
$$

Similarly

$$
\int_{S(0,R)} \frac{\partial}{\partial r} e^{i\pi j \cdot x / R} dS = \frac{1}{R} \int_{S(0,R)} \left( i\pi j \cdot \frac{x}{R} \right) e^{i\pi j \cdot x / R} dS
$$
\[ = R \int_{S(0,1)} (i\pi \cdot x) e^{i\pi j x} dS = R \int_{S(0,1)} i\pi |j| x e^{i\pi |j| x} dS \]

\[ = R i\pi |j| \int_{-1}^{1} x e^{i\pi |j| x} 2\pi dx_1 = 4\pi R \cos(\pi |j|) - \frac{4R}{|j|} \sin(\pi |j|), \quad j \neq 0. \]

Recalling that \( \varphi_j(x) = (2R)^{-3/2} \exp \left( -i\pi j \cdot \frac{x}{R} \right) \) for \( n = 3 \), this results to

\[ \hat{K}(j) = \frac{R^2}{\pi^2 |j|^2 - R^2} \left[ (2R)^{-3/2} \left( 1 - e^{iR} \left( \cos(\pi |j|) - i\frac{R}{|j|} \sin(\pi |j|) \right) \right) \right], \quad j \neq 0, \]

\[ \hat{K}(0) = -(2R)^{-3/2} \left[ 1 - e^{iR} (1 - Ri) \right]. \]

For \( \pi |j| = R \) we obtain

\[ \hat{K}(j) = -i2^{-5/2} R^{-1/2} (1 - e^{iR} R^{-1} \sin R) \]

by the L'Hospital rule or directly noticing that

\[ \Phi(|x|) = \frac{1}{8\pi i} (\Delta + 1) e^{i|x|} \quad \text{for} \quad n = 3. \]

Clearly \( |\hat{K}(j)| \leq c |j|^{-2} \) \( (j \neq 0) \) in the case of cutting (10), \( n = 3. \)

**Cutting (10), \( n = 2. \)**

\[ \int_{S(0,R)} e^{i\pi j x/R} ds = R \int_{S(0,1)} e^{i\pi j x} ds = R \int_{S(0,1)} e^{ix|j|x} ds \]

\[ = 2R \int_{-1}^{1} e^{i\pi |j|x} (1 - x^2)^{-1/2} dx_1 = 4R \int_{0}^{1} \cos(\pi |j| x_1) (1 - x^2)^{-1/2} dx_1 \]

\[ = 2\pi R J_0(\pi |j|) \]

and similarly

\[ \int_{S(0,R)} \frac{\partial}{\partial r} e^{i\pi j x/R} ds = -2\pi^2 |j| J_2(\pi |j|) \]

(see [1], formulae 9.1.20 and 9.1.28 for the Bessel functions \( J_0 \) and \( J_1 \)). This results to

\[ \hat{K}(j) = \frac{R^2}{\pi^2 |j|^2 - R^2} (2R)^{-1} \left\{ 1 + \frac{1}{2} i\pi |j| J_1(\pi |j|) H_0^0(R) - R J_0(\pi |j|) H_1^0(R) \right\} \]

for \( \pi |j| \neq R, \quad j \neq 0, \)

\[ \hat{K}(0) = -(2R)^{-1} - \pi \Phi'(R) = -(2R)^{-1} + \frac{\pi i}{4} H_1^{(1)}(R), \]

\[ \hat{K}(j) = \frac{1}{8\pi R} \left[ J_0(R) H_0^{(1)}(R) + J_1(R) H_1^{(1)}(R) \right] \quad \text{for} \quad \pi |j| = R. \]

Since \( J_\nu(r) \sim \sqrt{2/(\pi r)} \cos(r - \nu \pi - \frac{1}{2} \pi) \) as \( r \to \infty \) (see [1], formula 9.2.1), we have

\[ |\hat{K}(j)| \leq c |j|^{-3/2} \] \( (j \neq 0) \) in the case of cutting (10), \( n = 2. \)
Cutting (11), $n = 2$ or $n = 3$. Using the soft cutting (11) we obtain from (28)

$$
\tilde{K}(j) = \lambda_j \left\{ (2R)^{-n/2} - \int_{G_R} \left( 2\nabla \Phi \cdot \nabla \psi + \Phi \Delta \psi \right) \varphi_{-j} dx \right\} = \lambda_j \{ (2R)^{-n/2} - \chi(j) \}
$$

where the function

$$
\chi(x) := 2\nabla \Phi(|x|) \cdot \nabla \psi(|x|) + \Phi(|x|) \Delta \psi(|x|) = 2\Phi'(r) \psi'(r) + \Phi(r) \psi''(r) + 2r^{-1} \psi'(r)
$$

is $C^\infty$-smooth and supported on the annulus $2\varrho \leq r \leq R$ (see (12)). Therefore $|\tilde{\chi}(j)| \leq c_p |j|^{-p}$ ($0 \neq j \in \mathbb{Z}^n$) with any $p > 0$, and $|\tilde{K}(j)| \leq c |\lambda_j| \leq c' |j|^{-2}$ ($j \neq 0$).

Approximating $\chi$ by $Q_{M\lambda} \chi$, $M \sim N^\Theta$, $0 < \Theta \leq 1$, we have

$$
\max_{x \in G_R} |\chi(x) - (Q_{M\lambda} \chi)(x)| \leq c\|\chi - Q_{M\lambda} \chi\|_2 \leq c_p M^{-p}\|\chi\|_{p+2}
$$

with any $p > 0$. The computation of $(Q_{M\lambda} \chi)(j)$, $j \in \mathbb{Z}_M^n$, from grid values of $\chi$ by FFT costs $\mathcal{O}(M^n \log M)$ arithmetical operations. Thus, we have a cheap way to compute the approximations

$$
\hat{K}_N(j) = \frac{R^2}{\pi^2 |j|^2 - R^2} \left\{ (2R)^{-n/2} - (Q_{M\lambda} \chi)(j) \right\} \quad \text{for } j \in \mathbb{Z}_M^n,
$$

$$
\hat{K}_N(j) = \frac{R^2}{\pi^2 |j|^2 - R^2} (2R)^{-n/2} \quad \text{for } j \in \mathbb{Z}_M^n \setminus \mathbb{Z}_M^n
$$

to $\hat{K}(j)$ of a high accuracy:

$$
\max_{j \in \mathbb{Z}_N^n} |\hat{K}_N(j) - \hat{K}(j)| \leq c_p N^{-\Theta p}\|\chi\|_{p+2} \quad \text{with any } p > 0. \tag{29}
$$

There is also a possibility for an exact expression of $\tilde{\chi}(j)$ through integrals on $(2\varrho, R)$ from some smooth functions containing the multipliers $\psi'(r)$ and $\psi''(r)$.

Let us briefly discuss also the construction of cutting function $\psi$ satisfying (12). Take a function $\varphi \in C^\infty[0, \infty)$ such that $\varphi^{(k)}(0) = 0$ for all $k = 0, 1, 2, \ldots$, e.g. $\varphi(s) = e^{-\delta/s}$, $\delta > 0$. Then

$$
\psi(r) := \begin{cases} 
1 & 0 \leq r \leq 2\varrho \\
\frac{1}{c_0} \int_{2\varrho}^R \varphi(s - 2\varrho) \varphi(R - s) ds, & 2\varrho \leq r \leq R \\
0 & r \geq R
\end{cases}
$$

with

$$
c_0 = \int_{2\varrho}^R \varphi(s - 2\varrho) \varphi(R - s) ds
$$

satisfies (12). A fortune is that we need only derivatives $\psi'(r)$ and $\psi''(r)$, not $\psi(r)$ itself, and the derivatives of $\psi$ are available from the formula. The only integral defining $c_0$ can be approximated with a high accuracy by the trapezoidal rule since the integrand and all its derivatives vanish at the end points $2\varrho$ and $R$. If we put $\varphi(s) = s^m$ we have $c_0 = \frac{(m)!}{(2m)!} (R - 2\varrho)^{2m-1}$ and $\psi \in C^m[0, \infty)$. With $m$ sufficiently large, also this cutting function $\psi$ is suitable for our purposes.
References


