MATHEMATICS COURSE Y3, EXERCISE 9

week 46, 11,11 – 15.11

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GREE: CH 12 (Laplace) and  CH10 (Fourier-series)
KRE: CH 6 (Laplace) and CH10 (Fourier-series)

Maple: with(inttrans): # V.4  (readlib(laplace)  # V.3)
laplace, invlaplace, convert;
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Beginning of the week

1. Sketch the inverse Laplace transformations for the following functions and use shifting if it is necessary (L-transform of integration, the derivation of L-transform). Check the results by **convert**-command.

a) $F(s) = \frac{s^2+3}{s(s^2+4)}$, b) $F(s) = \frac{s+2}{s(s-1)^3}$, c) $F(s) = \frac{1}{s^3(s^2+2)}$

2. Show s-shifting theorem (hyperbolic functions in exp-form):

a) $L\{\cosh at \cos at\} = \frac{s^3}{s^4+4a^4}$,

b) $L\{\sinh at \cos at\} = \frac{a(s^2-2a^2)}{s^4+4a^4}$.

3. Solve the following integral equation by using L-transformation and the convolution theorem.

$y(t) = \sin t + \int_0^t y(\tau)\sin(t-\tau)d\tau$

4. Do the problems 3 and 5 at the end of the week at Exercise 8.:

$$y'' + 4y' + 13y = 20e^{-t}, \quad y(0) = 1, y'(0) = 3 \quad ,$$
$$\ddot{x} + 2\dot{x} = H(t-1), \quad x(0) = 0, \dot{x}(0) = 1.$$

5. Present the functions a) $1/(1-x)$, b) $1/(1-x^2)$, c) e^{kx} as sums of even or odd functions. Draw the graphs of these functions and their corresponding summations of the even or odd functions.

6. Let $f(x) = x, \quad x \in (-\pi, \pi], \quad f(x+2\pi) = f(x)$. Draw the pictures of f for a couple of periods. Present the Fourier-series of f and draw the pictures of some summations $s_1(x), s_2(x), s_3(x), \ldots$.

end of the week

1. Look at the circuit problem: $Ri' + (1/C)i = e'(t)$, where $i(0) = 0$ and $e(t)$ increases from 0 to e_0 at time ε $e(t) = \varepsilon, \ t > \varepsilon$.

Find the current $i(t;\varepsilon)$ (Gree s. 564 formula 22) and determine $\lim_{\varepsilon \to 0} i(t, \varepsilon)$, when $t = 0$ and $t > 0$ (formula 23). You can use the add command if it happens that `laplace/invlaplace` doesn't work.

2. The Laplace-transform of the periodic (period T) function f can be calculated by the formula:

$$F(s) = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t)dt$$

Write a maple function **lapjaks** (use a better name if you like), which implements the formula. You can do it by at least two ways: **lapjaks:=(f,T,s)-¿...** or `lapjaks:=(laus,t,T,s)->....` Note that in the previous case f has to be defined as a function while in the latter case f can be treated as a sentence (t as a symbol). In previous case the variable of the integration is a local parameter. But here it is not allowed (at least V.3.). Try the exercise for some known periodic functions like sin and cos. Determine the Laplace transforms of the following functions:

- a) "Saw-like function": $f(t) = \frac{k}{p}t, \quad 0 < t < p, \quad f(t+p) = f(t)$
- b)

$$f(t) = \begin{cases} \sin \omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$$

(continue $\frac{2\pi}{\omega}$-period).

3. Solve the damped oscillation problem

$$x'' + 2x' + x = \delta(t) - H(t - 2\pi), x(0) = 0, x'(0) = 0.$$

Here the outside force is an impulse at time $t = 0$ and -1 at $t = 2\pi$. Draw the solution figure at the interval $[0, 2\pi]$.

4. Damped harmonic oscillation:

$$y'' + 3y' + 2y = r(t)$$

where $r(t)$ satisfies:
a) $r_1(t) = 1, t \in [0,1]$, 0 otherwise, b) $r_2(t) = t, t \in [0,1]$, 0 otherwise, c) $r_3(t) = 10, t \in [0,5] \cup [10,15] \cup [20,25]$
Determine the solutions $y(t)$ for the above cases under the initial condition $y(0) = y'(0) = 0$. Draw the graphs of y and r into the same pictures.

Hint: Maple codes: `dsolve({dy,y(0)=0,D(y)(0)=0},y(t),method=laplace)`

5. Undamped forced oscillation:

$$y'' + \omega_0^2 y = K \sin \omega t$$

Find its solutions by the Laplace-transformations (by steps, not dsolve) for the following cases:
a) no resonance ($\omega \neq \omega_0$)
b) resonance ($\omega = \omega_0$).
Take the initial values $y(0) = y'(0) = 0$.

For symbols ω and ω_0, keep them in different forms. In b) case you need the command `convert()`.

6. Present the Fourier-series of the function

$$f(x) = \begin{cases} 100, & -4 < x < -2 \\ 0, & -2 \leq x \leq 4 \end{cases},$$

$$f(x+8) = f(x)$$

Draw the graphs of some partial summations.

Laplace-transforms

Given $f(t)$, $Lf = F$, $F(s) = \int_0^\infty f(t)e^{-st}dt$ **Notation** $u(t) = H(t)$=unit step function
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$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$
t^k	$\frac{k!}{s^{k+1}}$	e^{at}	$\frac{1}{s-a}$	$\cosh at$	$\frac{s}{s^2-a^2}$	$\sinh at$	$\frac{a}{s^2-a^2}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

$$(Lf')(s) = sF(s) - f(0), \quad (Lf'')(s) = s^2F(s) - sf(0) - f'(0),$$

$$(Lf^{(n)})(s) = s^n F(s) - s^{(n-1)} f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

$$L\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}F(s),\ \ L(f{*}g) = (Lf)(Lg), (f{*}g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = (g{*}f)(t)$$

$$L\{e^{at}f(t)\} = F(s-a),\quad L\{u(t-a)f(t-a)\} = e^{-as}F(s),\quad L\{\delta(t-a)\} = e^{-as}$$

Fourier-formulas

$f(x)$ is defined at the interval $(-L, L)$, $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$
$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,\quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$

Halp-range expansion: f is defined at the interval $(0, L)$

Sin-series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},\quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \qquad (HS_1, HS_2)$$

Cos-series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L},\ \ a_0 = \frac{1}{L}\int_0^L f(x)dx,\ \ a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \qquad (HC_1 - HC_3)$$