MATHEMATICS COURSE Y3, EXERCISE 8

week 45, 4,11-8,11

Beginning of the week

• Greenberg. CH12 p538->
• KRE. CH6 p261->
• TE: DYS-lecture in WWW (link: Oppikirjoja...), p56-60 Numerical methods.
• MATLAB instructions at Oppikirjoja, Ohjelmisto-oppaita and links Laplace transform tables at the last page of this exercise.
• Maple: with(linalg): Laplace, invlaplace, convert(..);

• Note: By computing the critical point you can linearize the system .

1. a) For what value p the integral $\int_0^\infty t^{-p} dt$ converges?

b) Show that $\int_0^\infty t^\alpha e^{-t} dt$ converges when $\alpha > -1$.

2. Piecewise continuous function is an exp-degree function if there exist constants K, c, T such that $|f(t)| \leq Ke^{ct}, \quad \forall t \geq T$ (please correct the printing errors in Greenberg).

a) Show that e^{t^2} is not an exp-degree function.

b) Is the derivative of an exp-degree function also an exp-degree function?

Hint: Study the function $f(t) = \sin(e^{t^2})$.

3. a) Determine the Laplace transforms of the functions $\sin^2 \omega t$ and $\cos^2 \omega t$.

Hint: When you get one answer the other one can be obtained by the help of the formula $\sin^2 + \cos^2 = 1$.

b) Determine $\mathbf{L}\{e^{\alpha t} \cos \omega t\}$ and check the result by using Laplace transformation table.

Hint: You can use partial integration or the equation $\cos\omega t = Ree^{i\omega t}$.

Try also Laplace transformation commands (**int**-command and **Laplace**-command) in **linalg** in Maple. Note that **int**-command can not always give the limitation value, it is sometimes embeded in the form of `int(f*exp(...),t=0..b)`, you can then find out the limitation value by hand calculations.

4. Determine the Laplace transforms of the following functions:

$$f(t) = \begin{cases} 1, & t \in [1,2] \\ 0, & t \notin [1,2] \end{cases}$$

and

$$g(t) = \begin{cases} t, & t \in [0,1] \\ 0, & t > 1 \end{cases} .$$

Check your results by **int**-command in Maple. Try also **Plot(He)**. You can give immediately their Laplace transforms after small modifications.

5. Use Laplace-transform of the derivative (also higher derivative if needed) in the following Laplace-transform problems **a)** $t\cos t$, **b)** te^{-5t}, **c)** $\cosh^2 t$

6. The convolution of functions f and g is defined as

$$f * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Determine the inverses of the Laplace transforms of the following functions by using the convolution theorem $(f*g) = (Lf)(Lg)$:
a) $\frac{1}{(s-a)^2}$ **b)** $\frac{1}{s^2(s^2+\omega^2)}$ **c)** $\frac{s}{(s^2+\omega^2)^2}$

End of the week

1. One step Euler method for the homogeneous system $x' = Ax$ can be used in MATLAB as follows:

```
t0=0;x0=[..;..];h=..;t1=t0+h;x1=x0+h*A*x0
```

To handle the time dependent problem you can do the following iteration:

```
t1=t0+h;x1=x0+h*A*x0;hold on;plot([t0,t1]',[x0,x1]'),t0=t1;x0=x1
```

Solve the stiff problem on the interval $[0, 0.1]$ (exercise 7, at the beginning of the week, problem 5) and draw the solution graph by **linsys**. Repeat this exercise on the intervals $[0, 1]$ and $[0, 10]$.

2. "Backward Euler BE" method is an implicit method whose schematic is $x_{n+1} = x_n + hf(t_{n+1}, x_{n+1})$. Let's look at a linear homogeneous system $x' = Ax$.

a) Show that the BE formula can be written as $x_{n+1} = (I - hA)^{-1}x_n$.

b) Why the matrix $I - hA$ is always invertible if the problem is stiff?

c) Do the previous exercise by using BE method and have a look the solution graphs.

3. Solve the initial value problem $y'' + 4y' + 13y = 20e^{-t}$, $y(0) = 1, y'(0) = 3$ by using Laplace transformation. The related commands in Maple are **laplace, invlaplace, solve, convert (F,parfrac,s)** etc.

Solution: $y(t) = 2e^{-t} + e^{-2t}\sin 3t - e^{-2t}\cos 3t$.
For checking: `dsolve({diffyht,y(0)=1,D(y)(0)=3},y(t));`

4. Express the following functions by Heaviside function H. and draw the function graphs. Remember that there exists the Heaviside command in Maple.

5. Solve the initial value problem $\ddot{x} + 2\dot{x} = H(t-1), \quad x(0) = 0, \dot{x}(0) = 1.$

6. Solve the differential equation system

$$\begin{cases} y_1' + y_2' = -\sin t \\ y_2' + y_3' = \cos t + (1/10)\exp(-t/10) \\ y_3' + y_1' = (1/10)\exp(-t/10) - \sin t \end{cases},$$

where the intial conditions are $y_1(0) = 1, y_2(0) = 0, y_3(0) = 0$. By using Laplace transformation you can do this exercise by Maple (dsolve, not directly) or/and by hand. Draw the phase space curve on $t \in [0, 8\pi]$ by the space-curve function. The system is not autonomous. Compare the picture with those of autonomous systems.

Laplace transform

Definition: Given $f(t)$, $Lf = F$, $F(s) = \int_0^\infty f(t)e^{-st}dt$.
Notation $u(t) = H(t)$=unit step function.

$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$
t^k	$\frac{k!}{s^{k+1}}$	e^{at}	$\frac{1}{s-a}$	$\cosh at$	$\frac{s}{s^2-a^2}$	$\sinh at$	$\frac{a}{s^2-a^2}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

$$(Lf')(s) = sF(s) - f(0), \quad (Lf'')(s) = s^2F(s) - sf(0) - f'(0),$$

$$(Lf^{(n)})(s) = s^nF(s) - s^{(n-1)}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

$$L\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}F(s), \;\; L(f*g) = (Lf)(Lg), (f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = (g*f)(t)$$

$$L\{e^{at}f(t)\} = F(s-a), \quad L\{u(t-a)f(t-a)\} = e^{-as}F(s), \quad L\{\delta(t-a)\} = e^{-as}$$

$\delta(t-a)$ presents *Dirac's* δ function.