MATHEMATICS COURSE Y3, EXERCISE 7

week 44, 28,10-1,11

Beginning of the week

• Greenberg. CH20 p1034-> numerical methods for DE.

• Note: By computing the critical point you can linearize the system .

1. Solve the initial problem $x' = t^2 - x, \quad x(0) = 2$ by using Euler method at the interval $[0, 0.2]$ with the step size a.) $h = 0.2$ (only one step). b.) $h = 0.1$ (two steps).

Compare the numerical solutions with the analytical solution at the point $t = 0.2$. Does the error behave as you expected when the step size is smaller (you don't need computer to do that)?

2. Solve the initial value problem $y' + 5t^4y^2 = 0, \quad y(0) = 1$ by using Euler method [1] with the step size $h = 0.2$ for 10 steps. The following Maple code can help to solve this problem too.

```
t:='t':y:='y':  # Security reason: release the values.
f:=(t,y)->-5*t^4*y^2; # The function which defines the diff. equation.
h:=0.2:N:=10;
t[0]:=0: y[0]:=1;
for n from 0 to N do
  y[n+1]:= ... ;
  t[n+1]:= ... ;
od;
```

Compute the analytical solution of this equation and compare it with its Euler solution. (At the beginning of this week it is enough to calculate it by hand when N is small. At the end of this week you 'll do it by Maple).

3. Use the 5th degree Taylor method [2] to solve the equation $x' = x + e^t + tx, \quad x(1) = 2$ with $h = 0.1$ for two steps (the derivative isn't difficult to compute. However, you

[1] $y_{n+1} = y_n + hf(t_n, y_n), \quad y(t_0) = y_0, \quad \text{GKV} = O(h)$

[2] $y(t+h) = y(t) + hy'(t) + ... + \frac{h^p y^{(p)}}{p!} + O(h^{p+1})$, where $y'(t) = f(t, y(t)), y''(t) = \frac{d}{dt}f(t, y(t))$, y'(t) $f(t, y(t))$

can still use Maple to compute it):

```
f:=(t,x)->exp(t)+x+t*x; d1f:=diff(f(t,x(t)),t);
d1f:=subs(diff(x(t),t)=f(t,x(t)),d1f));   # etc.
collect(df1,exp(t)); collect(",x(t));
### "COLLECT" is a useful command for collecting the coefficients.
```

4. Apply the one step Runge Kutta method into the initial value problem $x' = -\lambda x, \quad x(0) = 1$ and show that the local error is $O(h^5)$.

5. Let's look the linear system $x' = Ax$, where $A = \begin{bmatrix} 998 & 1998 \\ -999 & -1999 \end{bmatrix}$

You can get the following MATLAB-output:

```
>> [V,D]=eig(A)
V =                              D =                   [V(:,1)/V(1,1),(V(:,2)/V(1,2)]

   0.8944   -0.7071              -1          0             1.0000     1.0000
  -0.4472    0.7071               0      -1000            -0.5000    -1.0000
```

<u>Stiffness</u>: The system, whose eigenvalues of its Jacobian matrix are negative and the ratio of its maximal eigenvalue to its minimal eigenvalue (absolute values) is very big, is called stiff system (see the above problem). Find the analytical solution of the system $x' = Ax, x(0) = [1, 1]$ and compute its numerical solution also by using Euler method with the step size $h = 0.1$ starting from the initial point. You can find that the there exist problems on the numerical solution. The reason is because of the stiffness.

6. Look at the Lorenz system:

$$\begin{cases} x' = 10(y - z) \\ y' = x(28 - z) - y \\ z' = xy - (8/3)z \end{cases} .$$

Determine its equilibrium points, i.e. its critical points. You can find that the eigenvalues of the linearized system (i.e. Jacob matrix) are the equilibrium points.

End of the weekend

1. Try to solve problem 1 at the beginning of this week by the following way: Look at the homepage of this course at **Ohjelmakoodeja, maple-koodeja**. There you can find the definition of Euler-function. Copy it into your own Maple directory and use it for making the pictures and output-tables. Solve the problem on the interval [0,5] with step sizes $m = 3,\ 6,\ 12,\ 24$.

Find its analytical solution also. Draw the analytical solution curve and the numerical solution curve (with step size $m = 12,\ 24$) into the same picture. Think why the solutions explode when $m = 3$ and $m = 6$.

HINT: You can use the command **dsolve** (Maple) to solve the initial value problem by `dsolve({dyht,y(0)=1},y(t));` Drawing goes this way:

```
tarkka:=plot(rhs(ratk),t=0..5,color=red):
e12:=Euler(...,12);p12:=plot([e12]):...
display({tarkka,p12,p24});
```

You can do the following problem if you are interested in it: Try **RK4**-function. (it is at the same web site as Euler-function).

2. Do the previous problem on MATLAB by using Euler-function and RK4-function [3]

The information about Euler and RK4-functions can be obtained from the web at 'Ohjelmakoodeja". (In Windows, the left button of the mouse is used for pasting. Also you can use CTRL-C to mark the object and then click CTRL-V to paste the object). Another way to do it is to invoke the editor and copy the following codes:

```
hold on,grid,m=3;
[T,Y]=meuler('f6',a,b,ya,m);plot(T,Y);[T',Y'],if m<=30,plot(T,Y,'ow');end;m=2*m
```

To draw the analytical solution curves into the same picture (previous problem) you can do the following:

```
t=linspace(0,5);y=1./(t.^ 5+1);plot(t,y)
```

3. The classical victim-hunter-problem ('foxes' and 'rabbrabitsits'-model of Volterra-Lotka) is described as follows: In an area there exist two different kinds of animals.

[3]

$$\begin{cases} k_1 = hf(t_n, x_n) \\ k_2 = hf(t_n + \frac{h}{2}, x_n + \frac{k_1}{2}) \\ k_3 = hf(t_n + \frac{h}{2}, x_n + \frac{k_2}{2}) \\ k_4 = hf(t_n + h, x_n + k_3) \\ x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{cases}$$

GKV$= O(h^4)$

The amounts of them are r and f (f= amount of foxes, r= amount of rabbits). Let's suppose that the relative change speed of the rabbit amount is $\frac{r'}{r} = 2 - \alpha f$ and the relative change speed of the fox amount is $\frac{f'}{f} = -1 + \alpha r$. (Here r presents the amount of the victim animals and without the enemies f would increase exponentially. f represents the amount of hunters, which are living on eating r, f will die off if r is 0). Write the system as an m-file (then you can use MATLAB). Study the dynamical properties of the system when $\alpha = 0.01$.
a) Calculate the analytical solution when $r_0 = 300, \quad f_0 = 150$. Draw both the time dependent and the phase plane pictures. Note that the solution is periodic. What is the length of the period? Try also **ode23** and **ode45** solvers.
b) Draw the phase plane pictures (remember MATLAB command **hold on**) with different initial values $r(0) = r_0, \quad f(0) = f_0$. Vary the values from some tens to several thousands.
c) Find the equilibrium points of the system and describe their types (linearize the system).

Note: Odedemo (MATLAB) is related to the same topic. Here comes a surprise: ode23 can give more accurate solutions.

Stability

The equation is said to be stable at a point (t, y) if $f_y(t, y) < 0$. Numerical method is said to be stable if the total error is bounded. In Euler method this means that

$$-2 < hf_y(t, y) < 0.$$

4. Look at the equation $y' = -2\alpha(t-1)y$. Its general solution is $y = C\exp(-\alpha(t-1)^2)$. Show that it is stable/unstable by looking at the solution graph and its derivative values. Draw its solution collection (assume $\alpha = 5$).

Using the above method to study the solution of this problem with different step sizes. Draw the solution curves and figure out its stability properties. Try to get those information from the graphs as much as possible. The suitable iterative parameters might be $h = 0.2$ on $[1.4\ 5]$, $y(1) = 1$. They are suitable to MATLAB and Maple. Perhaps MATLAB is more friendly and interactively (matter of taste perhaps).
5. (Lorentz attractor) Using **ode25** or **ode45**-function (MATLAB) to draw the phase space pictures of the problem 6 at the beginning of this week. The view point can be changed by using **view** command. Draw suitable projections of these pictures into some subspaces and try to figure out the stabilities/instabilities by eigenvalues and eigenvectors(linearized system). Think also how trajectories behave.

Note: There is a function linalg(Jacobian) in Maple. You can use it to make the Jacobian matrix.

6. Try some other problems which you haven't done (for instance pendulum etc.) with the above methods .