MATHEMATICS COURSE Y3, EXERCISE 6

week 43, 21-25,10

Beginning of the week

- Greenberg . p183 - 189.
- Kreyszig 7^{th} ed. p158-183 .
- Eirola: DYT-textbook.

1. Compute e^A for $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. The following Maple commands can help you to solve this problem:

$> A := matrix(3, 3, [2, 1, 0, 1, 2, 1, 0, 1, 2]) :$
$> eigenvals(A);$

$2, 2 + \sqrt{2}, 2 - \sqrt{2}$

$> \ ev := eigenvects(A);$

```
[2, 1, {[ -1, 0, 1 ]}],
                2                              2
  [RootOf(_Z  - 4 _Z + 2), 1, {[ 1, RootOf(_Z  - 4 _Z + 2) - 2, 1 ]}]
```

The command **eigenvects** or **eigenvals** gives the result which is related to the roots of a second order equation (RootOf). You can try the following commands in order to get further results:

```
> map(allvalues,ev[2]);
       1/2       1/2            1/2                 1/2
  [2 + 2   , 2 - 2   , 1, {[ 1, 2   , 1 ]}, {[ 1, - 2   , 1 ]}]
#  You have to know how to read this output.
> eigenvects(convert(op(A),float));
      [.5857864374, 1, {[ .5000000000, -.7071067813, .5000000001 ]}],
```

```
                                                 -9
[2.000000000, 1, {[ -.7071067808, -.14*10   , .7071067812 ]}],
[3.414213564, 1, {[ .4999999998, .7071067814, .5000000002 ]}]
```

You can check your results by the exponential function in Maple.

2. Calculate e^{At} for the following matrices:

a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ c) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Solutions: b)$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ c) $e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

3. Free damped vibration:

$$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0.$$

Determine the types of the critical point (0,0) in the following cases:
a) no damping: $c = 0$. b) subdamping: $c^2 < 4mk$. c) critical damping: $c^2 = 4mk$.
d) overdamping: $c^2 > 4mk$.

Hint: You have already written this second equation into a first order equation (Exercise 5, problem 3). To determine the types of the critical point you could solve the equation and look at the signs of the roots. Please check the Eirola's lecture notes.

4. Solve $y' = Ay$, where $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$, by seeking the eigenvalue λ and its corresponding eigenvector x. Find the other related eigenvector u of λ which satisfies $(A - \lambda I)u = x$. (Remember that you can obtain the solution from the formula $y_2 = te^{\lambda t}x + e^{\lambda t}u$. See your lecture note)

5. Determine and characterize the equilibrium points of the system

$$\begin{cases} \dot{x} = x + 6y + 11 \\ \dot{y} = 5x + 2y - 1 \end{cases}.$$

6. Determine the equilibrium points of the system

$$\begin{cases} \dot{x} = x - x^2 + 2y + 3 \\ \dot{y} = -x + 2y \end{cases}.$$

Linearize the system w.r.t. the equilibrium points and characterize their types.

End of the weekend

1. Compute e^{At} for the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Perform your computational procedures by definitions. You can calculate matrix powers in Maple and MATLAB. The **exponential** (MAPLE) and **expm** (MATLAB) are suitable commands for checking the results. Remember that a symbol t is not allowed to be an argument in the command **expm** in MATLAB.

2. Solve the initial value problem $y' = Ay$, $y(0) = (0, 1, -1)$, where A is the same as the previous problem. You can use the exp-function. Draw the solution curve w.r.t. the time and the 'phase space graph'. The following is useful for drawing the phase space graph.

```
with(plots):
spacecurve([cos(t),sin(t),t]. t=0..4*Pi);
```

3. A linear constant coefficient system can be solved by the expm-function in MATLAB. You can find the following codes from the www site
(http://www.math.hut.fi/teaching/y3/kirjallisuus.html)
under **Ohjelmistoja**. Copy these codes into the file linsys.m under your working directory.

```
function [t,y] = linsys(A,x0,T,siz)
% First: A=[1 0 0;1 3 0;0 1 1];x0=[1;2;3];
%       [t,y]=linsys(A,x0,2);
%       plot(t,y)                          % time dependence
%       plot(y(:,1),y(:,2))                % phase plane (1,2)
%       plot3(y(:,1),y(:,2),y(:,3))        % phase space (3-d)
if nargin < 4, siz = 100; end;
t = linspace(0,T,siz);
m=size(A);m=m(1);
y = zeros(m,siz);
for i=1:siz,
y(:,i) = expm(A * t(i)) * x0;
end;
y=y';
```

Solve the previous problem by using the above codes. Draw several phase space curves. You can also draw y_1y_2--, y_1y_3--, y_2y_3 planes as projections (you can get the help by the command **help linsys**). You will notice that in MATLAB you can draw the above pictures into the same graphical window by the command **hold on**.

Maple-alternative: Define the function: `linsys:=(A,t,x0)->exponential(A,t)&*x0:`; The solution can be obtained even in its symbolic form. However, calculation and drawing don't go so smoothly as they do in MATLAB.

4. Solve the problem 4 at the beginning of this week (initial value problem) : $y(0) = (0, 1)$ by using the solution formula $y = e^{At}y(0)$.

Draw both time dependent graph and phase plane graph.

5. Solve the problem 5 at the beginning of this week by using Maple (or MATLAB) and draw its phase plane graph.

6. Solve the problem 6 at the beginning of this week by using Maple (or MATLAB) and draw its phase plane graph.