MATHEMATICS COURSE Y3, EXERCISE 5

week 42, 14-18,10

Beginning of the week

• Greenberg : 4.6 Application to systems, p183 - > Appendix A.
• Kreyszig 7^{th} ed. 4.2–4.4 p158-179 Syst. of DE., Phase Plane and stability. Some DE-topics are repeated in Chapters 1, 2 and 3. These are presumed to be known.
• Eirola: DYT-textbook.

1. Grossman II is also a suitable reference. Look at the initial value problem which describes the "damped Oscillator":

$$\begin{cases} x'' + 4x' + 13x = \cos t \\ x(0) = 0, \; x'(0) = 1. \end{cases}$$

Solve it by the "usual" way: 1) solve its homogeneous equation by using its characteristic equation, 2) seek its special solution y_p to find its general solution.

2. Draw the direction fields of the following differential equations. Solve them explicitly and draw their solution curves in the same picture (see the computer exercise). $a)$ $y' = 2y/t$ $b)$ $y' = -x/y$ $c)$ $9yy' + x = 0$

3. Write the 2nd order differential equation

$$my'' + cy' + ky = 0.$$

into the 1st order by assigning $y_1 = y, \; y_2 = y'$. The equation describes the free vibration of a damped spring. Solve the problem $y(0) = 0, \; y'(0) = 1$ by the matrix technique when $m = 1, \; c = 2, \; k = 0.75$.

Draw the pictures of its solution, its derivative, and its phase plane. The points on the phase plane are those of the form $(y_1(t), y_2(t))$ (t is a 'curve' parameter). You can check the result in Maple by the following way (for instance, actually there exist other easier ways also)

$y := ...$
$dyvp := m * diff(y, t, t) + c * diff(y, t) + k * y;$

$dyvp := subs(m = 1, c = 2, k = 0.75, dyvp);$
$subs(t = 0, y); subs(t = 0, diff(y, t));$
$plot(\{y, diff(y, t)\}, t = 0..5);$
$plot([y, diff(y, t)], t = 0..5);$

You can also try the command **dsolve**

$y :=' y'$ (for security reason)
$dy := subs(y = y(t), dyvp) = 0;$
$dsolve(\{dy, y(0) = 0, D(y)(0) = 1\}, y(x))$

4. Find the general solutions, the types of their critical point (0,0) and their stability properties for the following equations. Draw the pictures of their direction fields and solution curves.

$$a)\ \begin{cases} y_1' = 3y_1 - y_2 \\ y_2' = -y_1 + 3y_2 \end{cases} \qquad b)\ \begin{cases} y_1' = -4y_2 \\ y_2' = y_1 - 5y_2 \end{cases}$$

Hint: The essential command for the solutions in Maple is **eigenvects**. You can also try the exponential function for checking the "black box" **dsolve**. The direction fields and the approximated solution curves can be drawn by the command **DEplot** (with(DEtools)).

5. Do the same tasks as the previous one for the following equations:

$$a)\ \begin{cases} y_1' = -3y_1 + y_2 \\ y_2' = -4y_1 + 2y_2 \end{cases} \qquad b)\ \begin{cases} y_1' = -2y_1 + 2y_2 \\ y_2' = -2y_1 - 2y_2 \end{cases}.$$

6. Both tank A and tank B contain 50 l liquids. There are two pipes which connect A and B in such way that a liquid flows from A to B at the speed of 4 l/min and B to A at the speed of 1 l/min. Assume that liquids are mixed immediately. The clean water pumps into tank A with a speed of 3 l/min and the mixed liquid flows out of the tank B with the same speed of 3 l/min. Let's assume that at the beginning tank A contains 25 kg salt and tank B contains only fresh water. Determine how much tanks A and B contain the salt at different times (function of time). Draw the picture of the salt amounts contained at A and B at different times. i.e. $(x_1(t), x_2(t))$ where t is the time parameter.

```
3 l/min
----------          -----------
-->       |         |         |
--        |---------|         |
 |         4 l/min ->         |
 |        -----------         |
 |        |         |         |
 |     A  |         |    B    |
 |        |---------|         |
 |         <-- 1 l/min        |
 |        |---------|         ----
 |--------|         |----------> 3 l/min
```

End of the weekend

Try the command **with(DEtools)**. You can use for instance the function **DEplot** for drawing the direction fields and phase planes. It is also possible to draw the graph of the integral curve which goes through some given points. The curve can be obtained by using numerical method.

1. Draw graphs of the direction field and integral curve for the equation $y' = \sin t - y/2$ by the following way:

$dyht := diff(y(t), t) = \sin(t) - y/2;$
$Deplot(dyht, [y], t = 0..12, \{[0, 0], [0, 1], [0, -1]\}, y = -3..3, arrows = THIN), \%V.4(?)$
$Deplot(dyht, [t, y], t = 0..12, \{[0, 0], [0, 1], [0, -1]\}, y = -3..3, arrows = THIN), \%V.3(?)$

Draw the "same" pictures for the exercise 1 at the beginning of this week.

2. Draw the missing graphs for the exercises 4 and 5 at the beginning of this week (compare the pictures with those in the previous exercise).

3. Determine the general solution (current) in the following LRC-circuit when $E = 220 \sin t$.

```
      L=0.5 h          I2
   |-- UUUUU---|---->------|
   |           |           |
I1 ^                       |
   |           |           |
               <           <
   E           > R1=1      > R2=4
               <           <
   |           |           |
   |---------------| |------
                   C=1 F
```

4. Given a system

$$\begin{cases} y_1' = -y_1 + y_2 \\ y_2' = -y_1 - y_2. \end{cases}$$

Solve it by the command **dsolve**. Determine its eigenvalues. Draw the curve of its direction field which goes through the points $y = -3..3$ (i.e. points (0,-3)......(0,3))

Hint: Use the command **with(DEtools)** and some suitable functions like **DEplot**.

5. The solution of the nonhomogeneous system

$$\dot{x} = Ax + f,\ x(0) = \overline{x}_0$$

is

$$x(t) = e^{At}\overline{x}_0 + \int_0^t e^{A(t-u)} f(u)du.$$

(see your lecture note).
An integration for a vector function means the integration for each component of the function.

Solve the following equation by using the above result:

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Hint: It is recommended to diagonalize the matrix first and then apply the exp function to the diagonal matrix. You can also try the exponential function at the linalg package in Maple.

6. Look at the undamped pendulum problem: its equation is

$$\begin{cases} \Theta'(t) = v(t)/L \\ v'(t) = -g\sin(\Theta(t)). \end{cases}$$

Draw several phase plane curves and explain its physical meanings in the picture.