MATHEMATICS COURSE Y3, EXERCISE 4

week 41, 7-10,10

Beginning of the week

1. Find the matrices for the following matrix transformations:
a) $F: \ R^2 \to R^2$: projection to x_1-axis.
b) $F: \ R^3 \to R^2$: projection to $x_1 - x_2$-axis.
c) $F: R^2 \to R^2$: reflection w.r.t. x_2-axis.
d) $F: R^3 \to R^4: \ F(x) = \begin{pmatrix} x_1 - x_2 \\ 2x_2 - x_3 \\ 5x_3 - 2x_1 + x_2 \\ x_2 - 3x_1 \end{pmatrix}.$

2. Calculate the eigenvalues and eigenvectors of the "rotation matrix"

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

3. The eigenvalues of a matrix A are $\lambda_1 = 2, \ \lambda_2 = 4, \ \lambda_3 = 6$ and the corresponding eigenvectors are

$$x_1 = (1,0,1)^T, \ x_2 = (1,1,0)^T, \ x_3 = (0,1,1)^T.$$

Find A.

4. Let A be an $n \times n$ matrix.
a) Show that the eigenvalues of A and A^T are the same.
b) Assume that A is invertible. Let λ be an eigenvalue of A and x is its corresponding eigenvector. Show that $1/\lambda$ is an eigenvalue of the matrix A^{-1} and $y = AX$ is its corresponding eigenvector. Here $\lambda \neq 0$, why?
c) Show that if λ is the eigenvalue of A then λ^m is the eigenvalue of A^m.

5. Show that the eigenvalues of an upper (lower) triangular matrix are the diagonal elements of the matrix.

6. Find the eigenvalues and eigenvectors of the following matrices. Calculate at least one case by hand. You can do others by your computers using the command **eig-command** (MATLAB) or **eigenvects-command** (Maple). What matrix type are

these matrices belonged to? What can you say by simply looking them.

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{pmatrix}, \quad C = \begin{pmatrix} \sqrt{3}/3 & i\sqrt{6}/3 \\ -i\sqrt{6}/3 & -\sqrt{3}/3 \end{pmatrix}$$

End of the weekend

The web site of the MATLAB instructions is
http://www.math.hut.fi/teaching/y3/kirjallisuus.html.
You can get some MATLAB codes in Ohjelmakoodeja.

1. Let $A = \begin{pmatrix} k & 1 & 0 \\ 0 & k & 1 \\ 0 & 0 & m \end{pmatrix}$, where k and m are real and $k \neq m$. Find the eigenvalues and eigenvectors of A. Is A a defective matrix?

Hint: The eigenvalues can be found easily. The eigenvectors can be computed by using the command **charmat-command** or others in Maple. Please check your result by the command **eigenvects**.

2. MATLAB exercise: The spectrum of a matrix A is the set of all its eigenvalues on the complex plane.(n points, some of them might coincide. The spectral radius of A is defined by $r = \max(|\lambda_j|))$. The spectral circle is the unit circle with the radius r. The spectrum can reflect a lot of properties of the matrix. At the web site **Ohjelmakoodeja** you can find the command **z = spektri(A)** which draws the spectral circle for the matrix A on the complex plane.

Draw the spectra for symmetric, skew symmetric and orthogonal matrices.

Hint: Given an arbitrary matrix A, you can express it by a combination of some symmetric or skew symmetric matrices like A^TA, $A + A^T$, $A^T - A$ or some orthogonal matrices. To express it by the orthogonal matrices you can use the command **orth**, which generates the Gram-Schmidt orthogonal vectors for some given column vectors. Command **orth(rand(n,n))** is suitable for some 'random experiments", it generates a set of linear independent variables by "probability 1". Try 50×50 random matrices and compute their spectra.

3. In the lecture note it shows that the similarity transformations preserve the eigen-

values. Prove this statement by MATLAB for the matrices

$$A = \begin{pmatrix} 8 & 0 & 3 \\ 2 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

What about the eigenvectors? Try the following test:

$T = rand(3,3),\ eig(...);$

The command **Eig** works in MATLAB also. It gives the eigenvalues and the eigenspace by $[D, V] = eig(A)$ where D is the eigenvalue vectors and V the corresponding eigenvector space.

4. The following MATLAB commands define a matrix A whose sums of its columns are 1.

$>>\ A = [.5\ .25\ .25;\ .25\ .5\ .25;\ .25\ .25\ .5]$

$A =$

0.5000	0.2500	0.2500
0.2500	0.5000	0.2500
0.2500	0.2500	0.5000

$>>\ sum(A)$

$ans =$

1 1 1

The above matrix is stochastic matrix which represents the transform probabilities of the Markov process. Start from a vector u_0 in R^3 and do the iteration $u_0 = A * u_0$ you can get some ideas about the iterations for different starting points. Explain the result by representing the starting vectors by the eigenvectors (basis) multiplying A. You can check your observations by the command for instance $(A \wedge p)$. The stability of the vectors are caused by the same reason.

5. The following matrices are diagonalizable. If you don't know why then it is impossible for you to diagonalize the matrices.

$$a)\ A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad b) A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

6. Calculate A^{100} for A's in the previous exercise. Remember that those A's are diagonalizable. (MATLAB is suitable to do this task)

MATLAB code

```
function y = spektri(A)
d = eig(A);
r = max(abs(d));
th = linspace(0, 2*pi);
axlow = -r - r/10; axhigh = r + r/10;
plot(r*cos(th), r*sin(th));
hold on;
axis([axlow axhigh axlow axhigh]);
axis('square');
plot(real(d), imag(d), '*');
hold off;
```