MATHEMATICS COURSE Y3, EXERCISE 3

week 40, 30.9-4.10

Beginning of the week

1. a) Determine two vectors in R^3 which span the plane $x_1 - 2x_2 + 4x_3 = 0$.

b) Let P_2 be vector space which contains all polynomials with degrees at most 2. Are the polynomials $x - 2x^2$, $x^2 - 4x$ and $-7x + 8x^2$ linear dependent in P_2?

2. Determine the rank of the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & -1 & 3 \\ 1 & 2 & -1 & -1 & 4 \\ 2 & -1 & 1 & 0 & 0 \end{pmatrix}$. Also find out the bases for its column and row spaces. (column space = span of the column vectors)

Hint: To obtain the basis for the row space you can for instance use the Gauss row operations on the matrix and then pick up all non-zero rows (why ?). Similarly, the basis of the column space can be obtained from "the leading columns" of the Gauss or Gauss-jordan form (rref) of the matrix (why?). Gram-Schmidt helps to find the orthogonal basis if it is needed.

3. a) Show that if A is a square matrix, then its rows are linear independent if and only if its columns are linear independent.

b) Prove that if A is not a square matrix then either its rows or its columns are linear dependent.

4. Show that the determinant of an upper triangular matrix (or lower triangular matrix) is the product of its diagonal elements.

5. Determine, by calculating the determinant of a suitable matrix, weather the vectors (1, 2, 3, 4), (5, 6, 7, 8), (9, 10, 11, 12) and (13, 14, 15, 16) are linear independent or not. It is recommended to use the row operations which are related with the Gauss elimination.

6. A square matrix A is said to be nilpotent if $A^p = 0$ for some p. Show that if A is nilpotent then $I - A$ is invertible and $(I-A)^{-1} = I + A + A^2 + \cdots + A^{p-1}$.
Hint: You need only the definition of the inverse matrix.

End of the weekend

MATLAB is need in the following exercises. Its instructions can be found at the web site "//www.math.hut.fi/teaching/y3/kirjallisuus.html".

start: *use matlab* or
matlab (*Unix*) *Icon* (*Windows*)
help
help rref, *help rank*

1. [K] Given a 4×6 system $Ax = c$, where

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 2 & 4 & 1 & 2 \\ 0 & 2 & 1 & 4 & 3 & 1 \\ 1 & -3 & 0 & -4 & -2 & 1 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix}.$$

Drag its 4×4 unit submatrix by the command $[R, jb] = rref(A)$. Also pick up the columns from the original matrix which constitute the maximal linear independent set (the basis of the column space).

Make an augment matrix $Ac = [A, c]$ (MATLAB demands that c is a column vector) and check that your guess for the ranks of A and Ac coincides with your MATLAB result.

Note that the concept of the rank determines the number of the solutions (see GREE. Thm 4, p98)

Hint: It's very easy to pick up the submatrix from a bigger matrix in MATLAB. For instance, if you want to take the columns 2, 3 and 5 and then put them in a variable B you can simply give the command $B = A(:, [2\ 3\ 5])$ (In Maple $B = submatrix(A, 1..4, [2\ 3\ 5]);$).

2 [K] Let

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 2 \\ -1 & 2 & 4 & -1 & 0 \\ 5 & -2 & -12 & 5 & 4 \\ 1 & 1 & 0 & 3 & 2 \\ -4 & -1 & 4 & -10 & -6 \end{pmatrix}.$$

Determine its maximal linear independent rows by using MATLAB or Maple. Note that the command *rowspace* doesn't always seek the vectors whose span is the space, so it is not always suitable.

3. Let

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}.$$

Construct the matrix $A - \lambda I$ whose determinant is the characteristic polynomial. Insert 0's into the characteristic matrix and calculate the corresponding eigenvectors by the command *linsolve* or *rref*. Check your results by the commands *eigenvals* and *eigenvects* (compare them with *eig* command in MATLAB).
Hint:
$A := matrix(3, 3, [-2, 2, -3, 2, 1, -6, -1, -2, 0]);$
$Id := diag(1, 1, 1);$
$Cmat := A - lambda * Id;$ (compare *charmat*)
$Cpoly := ...$ (compare *charpoly*)
$lam := roots(...);$
$Cmat1 := subs(lambda = ..., Cmat); Cmat2 := ...$
gaussjord (or *rref*, *linsolve* $\rightarrow$ eigenvectors)
eigenvects (checking)

4 [K]. Given a linear mapping $F: \ R^2 \rightarrow R^2: \ F((1,1)) = (9, -3)$ and $F((2,1)) = (3, 6)$. Determine the mapping matrix (w.r.t. the natural basis) and find out the image of the point (1,-1) (actually computer is not necessary).

5. MATLAB exercise: You can use 'the house of Strang" to demonstrate the linear mappings. The following command might work:

$T = [0\ 0\ -1\ 6\ 13\ 12\ 12\ 3\ 3\ 6\ 6\ 0; 0\ 9\ 8\ 15\ 8\ 9\ 0\ 0\ 5\ 5\ 0\ 0]$
$plot(T(1, :), T(2, :))$

The above draws a picture whose x-coordinate and y-coordinate are the first column and the second row of the matrix.

Let A be a 2×2 matrix and F_A a linear mapping which is determined by A: the image y of a column vector x is obtained by the multiplication $y = Ax$. The image is a set of some points which are the collection of the column vectors in R^2 (as T above). So the corners of the house can be obtained by $S = AT$.

Let A be a rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Try the following:

$th = pi/4; A = [..., ...]; S = T;$
$>> S = A * S; plot(S(1,:), S(2,:))$ (iteration)

Try to predict how the house looks like under some mappings (try some). In order to save the space the matrices could be represented as

$[1\ 0; 0\ {-1}], [-1\ 0; 0\ 1]; [0\ 1; -1\ 0]; [0\ 1; 1\ 0]; [0.5\ 0; 0\ 0.5]; [2\ 0; 0\ 2]; [1\ 0; 1/3\ 2/3]; [1\ 1/4; 1/4\ 1].$

6 [K]. Determine the eigenvalues and eigenvectors of the following matrices. Find out the algebraic and geometrical degrees of each eigenvector (algebraic degree = degree of the root, geometrical degree = the dimension of the eigenvector space).

Which of the following matrices are defective (g-kl<a-kl)? Determine the basis of the space in each of the non-defective case by using eigenvectors.

Use *charpoly* to compute the characteristic polynomial and check your result by *eigenvects* command (you can also try the *eig*-command in MATLAB).

$$a)\ A = \begin{pmatrix} 8 & 6 & 6 \\ -3 & -1 & -3 \\ -6 & -6 & -4 \end{pmatrix} \quad b)\ B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad c)\ C = \begin{pmatrix} 109 & 87 & 81 \\ 87 & 73 & 63 \\ 81 & 63 & 61 \end{pmatrix}.$$