MATHEMATICS COURSE Y3, EXERCISE 2

week 39, 23.9-27.9

Beginning of the week

Gree. p3-61

1. Given a system of linear equations.

$$\begin{cases} 2x + y + z = 10 \\ 3x + y - z = 6 \\ x - 2y - 4z = -10 \end{cases}$$

a) Solve the linear equations by applying the Gauss Elimination. Try to learn how to handle this kind of problems by using coefficient matrices. Observe that we can also use the Gauss Elimination to transform the system into an upper part triangle and do the back-substitution steps (Gauss-Jordan reduction) from the bottom to the top.

b) Write the system as a vector equation whose left hand side is a linear combination of the vectors (or columns). Interpret the result of a) by vector theories.

2. Derive the solution set for each of the following systems, using the Gauss Elimination algorithm.

$$b) \begin{cases} x_1 - x_2 + 2x_3 + x_4 = -1 \\ 2x_1 + x_2 + x_3 - x_4 = 4 \\ x_1 + 2x_2 - x_3 - 2x_4 = 5 \\ x_1 + x_3 = 1 \end{cases}$$

$$c) \begin{cases} 2x_1 - x_2 - x_3 - 3x_4 = 0 \\ x_1 - x_2 + 4x_3 = 2 \end{cases}$$

Hint 1: The solution of c) is

$$\begin{cases} x_4 = \alpha_1 \\ x_3 = \alpha_2 \\ x_2 = -4 + 3\alpha_1 + 9\alpha_2 \\ x_1 = -2 + 3\alpha_1 + 5\alpha_2 \end{cases}$$

Hint 2: The purpose is to train your hand calculation skills. You can use Maple by the following steps:

$with(linalg);$
$A := matrix([[1,-1,2,1],[2,1,1,-1],[1,2,-1,-2],[1,0,1,0]]);$
$b := vector([-1,4,5,1]);$
$Ab := augment(A,b);$
$A1 := addrow(Ab,1,2,-2);\ \#\ rivi2 --> \ -2*rivi1 + rivi2$
$A2 := addrow(A1,1,3,-1);\ \#\ rivi3\ --> \ -1*rivi1 + rivi3$
...
$(?addrow)$

To check the result you can use the command **backsub**. You can also do the Gauss Elimination by simply giving the command **gausselim**. Note that the row operation can be done by MATLAB too, unfortunately the round-off error can lead to intolerable large errors in the solution. There is no gausselim-command or similar one in MATLAB. However **rref** command exists in both Maple and MATLAB which does the Gauss-Jordan reduction.

3. Can four linear equations in six unknowns have no solution? A unique solution? A one-, two-, three-, four-, five-, or six-parameter family of solutions? For each affirmative answer give one illustrative example.
Hint: Write the system into its reduced form.

4.
a) Show weather or not the vectors
(0, 0, 0, 1), (0, 0, 1, 1), (0, 1, 1, 1) and (1, 1, 1, 1) span R^4.

b) Show that the vectors $v_1 = (1,1,0)$, $v_2 = (0,1,1)$ and $v_3 = (1,0,1)$ form a basis of the space R^3. What is the coordinate of the vector $x = (1,2,3)$ in terms of this basis?

Note: The basis $\{v_1, v_2, v_3, v_4\}$ spans the space R^4 means that $\forall\ v \in R^4\ \exists\ x_1 ... x_4$ such that $v = x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4$.
Write vectors as column vectors and change the "column thinking" into the "row

thinking".(opposite to the problem 1 b)).
The basis of a space (here R^3) is a set of vectors which are a) linear independent and b) span the whole space. Since $dim(R^3) = 3$, three vectors form a basis if they satisfy the the conditions a) and b). (Think this way: if these 3 vectors span the space but they are not linear independent then...)

5. Show that vectors $e_1 = (2, 1, 3)$, $e_2 = (1, -2, 0)$, $e_3 = (6, 3, -5)$ form an orthogonal basis for R^3. What is the coordinate of the vector $u = (9, -2, 4)$ in terms of this basis?

6. Use the Gram-Schmidt formula in (1,1,0), (2,-1,1), (1,0,3) to obtain an orthonormal set.
Solution: $(1\sqrt{2}, 1/\sqrt{2}, 0), (3/\sqrt{22}, -3/\sqrt{22}, 2/\sqrt{22}), (-1/\sqrt{11}, 1/\sqrt{11}, 3/\sqrt{11})$

End of the weekend

The problem which is marked by **K** is expected to be solved beforehand like the ones at the beginning of the week. An non-K-exercise can be done afterward.

Maple instructions

The packages which are related with vector or matrix calculus are loaded by the command **with(linalg)**. (If you don't want the whole result to be printed you can use the mark **(:)**).

Matrix multiplication is represented by $A \ \&* \ B$. The command **evalm** is need for calculating the result, for example:

$A := matrix([[3, -1, 1], [-15, 6, -5], [5, -2, 2]]);$
$B := matrix([[2, 0, 1], [3, 1/2, 4], [1, -1, -5]]);$
$evalm(5 * A - 3 * B) :$
evalm(A & * B):

To have result evaluated the last two commands are needed. Also because of the mark **(:)** the content of the matrix is not able to be printed. To print the result you need the command **print(A)** or **op(A)**(op-"operand").

Some of the useful linalg functions are **addrow, augment, backsub, col, evalm, gausselim, GramSchmidt, innerprod, inverse, linsolve, matrix, op, print, row, rref, stack, submatrix, vector**.

1. Do the exercises 1 and 2 of the beginning of this week by using Maple commands **gausselim, backsub**. Try the commands **rref, linsolve** also. Make sure that you can read the solutions which are the outputs of both **gausselim** and **rref**.

Try the command **rref** on MATLAB.

Note: In MATLAB the matrix is written as follows:

$A = [2\ 1\ 1;\ 3\ 1\ \ -1\ ...];\ b = [10; 6; -10]; Ab = [A\ b];$
$help\ rref$

2. Let $m \times n$ system $Ax = b$ be given by random numbers. What do you think about the existence of the solutions and the number of the solutions (How many free parameters are there) if a) m = n = 20, b) m = 20, n = 25, c) m = 30, n = 20? Confirm your answers by the following way:

$with(linalg)$ # load the linalg package.
$A := randmatrix(20, 26);$
$?gausselim,\ ?backsub$
$evalf('\ ')$ simplify the output.

3. [K]
Kirchhof's current law: at any point of an electrical circuit the sum of the incoming currents must equal the sum of the outgoing currents (incoming flow = outgoing flow). Kirchhof's voltage law: the algebraic sum of the voltage rises around each closed loop of a circuit must equal zero. Apply the above laws into the following circuits, make the linear equations of the currents and solve the equations.

picture

4. [K]
Traffic net: The streets are indicated by arrows. The amounts of the traffic in one hour at the junction points (coming or outgoing) are marked also. The traffic which arrives at certain junction point has to leave at that point. We can build a set of equations to determine the unknown traffics x_1, x_2, x_3, x_4 inside the traffic net. Solve the equations. Show that there is an infinitely number of solutions for the equations. Some restrictions have to be put into the traffic net such as the traffic direction cannot be changed at the junction points and so on.

picture

5.[K]
Compute the inverse of the matrix

$$A = \begin{pmatrix} 371 & -76 & -40 \\ 36 & -7 & -4 \\ -176 & 36 & 19 \end{pmatrix}$$

by Gauss-Jordan algorithm. You can calculate the inverse by hand or by computers (Maple, MATLAB). By computers you can only use **rref** or **gaussjord** commands. (**inv** in MATLAB)

6. Study the Maple command **GramSchmidt** and use it to do the exercise 6 of the beginning of this week.