Helsinki University of Technology
Institute of Mathematics
Mat-1.3607 Stochastic Optimization, Fall 2006

Exercise 4
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1 Assume that the unit price of one of the assets is P-almost surely strictly positive for every $k = 0, \dots, K$ and show that the optimum value of the utility maximization problem

$$\begin{array}{ll} \underset{x \in \mathcal{N}^p}{\text{maximize}} & Eu(s_K(\xi) \cdot x_K(\xi)) \\ \text{subject to} & s_0(\xi) \cdot x_0(\xi) \le V_0, \\ & s_k(\xi) \cdot (x_k(\xi) - x_{k-1}(\xi)) \le 0, \quad k = 1, \dots, K, \\ & P\text{-a.s.} \end{array}$$

remains unchanged if we replace the inequality constraints by equalities. How would you interpret the possibility of allowing inequalities instead of equalities?

2 (a) Show that if the price process s does not satisfy the no-arbitrage condition then the optimum value in

$$\begin{array}{ll} \underset{x \in \mathcal{N}}{\text{maximize}} & E[s_K(\xi) \cdot x_K(\xi)] \\ \text{subject to} & s_0(\xi) \cdot x_0(\xi) \le 0, \\ & s_k(\xi) \cdot (x_k(\xi) - x_{k-1}(\xi)) \le 0, \quad k = 1, \dots, K, \\ & s_K(\xi) \cdot x_K(\xi) \ge 0, \\ & P\text{-a.s.} \end{array}$$

is $+\infty$.

(b) Assume that s satisfies the no-arbitrage condition and that the unit price of one of the assets is P-almost surely strictly positive for every $k = 0, \dots, K$. Show that if $x \in \mathcal{N}^p$ satisfies the above constraints then it satisfies the constraints as equalities.

3 A function $f : X \to \overline{\mathbb{R}}$ is said to be *sublinear* if it is

- *positively homogeneous:* $0 \in \operatorname{dom} f$ and $f(\alpha x) = \alpha f(x)$ for every $x \in X$ and $\alpha > 0$
- *subadditive:* $f(x_1 + x_2) \le f(x_1) + f(x_2)$ for every $x_1, x_2 \in X$

(a) Show that f is sublinear iff it is convex and positively homogeneous.

(b) Show that $F \mapsto \overline{\pi}(F)$ is a sublinear function on the space of measurable functions.

(c) Show that the set of replicable contingent claims is a linear space and that $\overline{\pi}$ is a linear function on it.

(d) Let F be a futures contract, F_c a European call option and F_p a European put option, all with the same strike price. Show that $\overline{\pi}(F_p) \le \overline{\pi}(F_c) + \overline{\pi}(-F)$ and that if at least two of the claims is replicable then the "put-call parity" $\overline{\pi}(F_p) = \overline{\pi}(F_c) - \overline{\pi}(F)$ holds.

4 Consider a market where trading costs are given in terms of a nonlinear price process in the sense of Definition 2.6 and formulate an optimization problem giving the "super-hedging price" $\overline{\pi}(F)$ of a contingent claim F. Show that $\overline{\pi}$ is a convex function on the space of measurable functions.