1 Recall that, by definition, an $\mathbb{R}^m$-valued function x_k is measurable with respect to a σ-field $\mathcal{F}' \subset \mathcal{F}$ iff

$$x_k^{-1}(C) := \{\xi \in \Xi \,|\, x_k(\xi) \in C\} \in \mathcal{F}'$$

for every open $C \subset \mathbb{R}^m$. Verify that (as claimed on page 24) x_k is $\mathcal{F}_k$-measurable iff it is $\mathcal{F}$ measurable and independent of $(\xi_{k+1}, \ldots, \xi_K)$.

2 Let $\xi_1, \ldots, \xi_k$ be random variables and $\mathcal{F}_k = \sigma(\{\xi_1, \ldots, \xi_k\})$ (i.e. smallest sigma-algebra generated $\{\xi_1, \ldots, \xi_k\}$). Then $x \in \mathcal{L}^0(\mathcal{F}_k)$ if and only if there exists a measurable $f : R^k \to R$ such that $x(\omega) = f(\xi_1(\omega), \ldots, \xi_k(\omega))$ (here ω is an element from the probability space).

Prove the claim in the case $k = 1$.

(Hint: Consider only simple functions (i.e. piece-wise constant functions).)

Solution: The implication $\Leftarrow$ is straight-forward. Let us prove the other direction:
Assume $k = 1$. Take $x \in \mathcal{L}^0(\mathcal{F}_1)$ and assume it is of the form

$$x(\omega) = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(\omega),$$

where a_i are constants, $A_i \in \mathcal{F}_1$ and $\mathbb{1}_{A_i}(\omega) = \begin{cases} 1 & ,\omega \in A_i \\ 0 & ,\text{otherwise.} \end{cases}$
Because $A_i \in \mathcal{F}_1$, there exists $B_i \in \mathcal{B}(R)$ s.t. $A_i = \xi_1^{-1}(B_i)$. Thus $x(\omega)$ can be written in the form

$$x(\omega) = \sum_{i=1}^{n} a_i \mathbb{1}_{B_i}(\xi_1(\omega)) =: f(\xi_1(\omega).$$

Now, take $x \in \mathcal{L}^0(\mathcal{F}_1)$. Because it possible to approximate an arbitrary measurable function point-wise by simple functions, there exists a sequence $x_n(\omega) = f_n(\xi_1(\omega))$ such that for every ω we have $f_n(\xi(\omega)) \to x(\omega)$. As point-wise limits preserve measurability we have $f_n(\xi_1(\omega)) \to f(\xi_1(\omega)) = x(\omega)$ where f is measurable.

3 Consider the reduced formulation of a stochastic control model in Section 2.2. Give conditions under which

$$f(u, \xi) = \varphi(F(u(\xi), \xi), u(\xi), \xi)$$

is a normal integrand. What extra conditions would make f a convex normal integrand?

4 Apply Theorem 2.4 to the examples of Section 2.2