1 Let

$$f(x,\xi) = \frac{1}{2}x \cdot A(\xi)x + a(\xi) \cdot x,$$

where $A : \Xi \to \mathbb{R}^{n\times n}$ and $a : \Xi \to \mathbb{R}^d$ are measurable. Show that

(a) f is a normal integrand.

(b) if $a \in L^q$ and $A(\xi)$ is positive semidefinite for every $\xi \in \Xi$, then Ef is σ_p-lsc on L^p.

(c) if $a \in L^2$ and there is an $\alpha > 0$ such that $x \cdot A(\xi)x \geq \alpha|x|^2$ for every $x \in \mathbb{R}^n$ and $\xi \in \Xi$, then $\operatorname{argmin} Ef$ is nonempty and σ_2-compact in L^2.

(d) if $A \in L^\infty(\Xi, \mathcal{F}, P; \mathbb{R}^{n\times n})$ and $a \in L^q$ then Ef is τ_p-continuous on L^p for all $p \geq 2$.

2 Show that if f_1 and f_2 are normal integrands such that Ef_1 or Ef_2 is proper, then

$$E(f_1 + f_2) = Ef_1 + Ef_2.$$

3 A probability measure Q on $(\Xi, \mathcal{F})$ is said to be *absolutely continuous* with respect to P, denoted $Q \ll P$, if for every $A \in \mathcal{F}$

$$P(A) = 0 \implies Q(A) = 0.$$

Show that for any random variable x,

$$\operatorname{ess\,sup}^P x = \sup_{Q \ll P} E^Q x.$$

4 Show that if f is a nonnegative convex normal integrand, then the function

$$F(x) = \operatorname{ess\,sup}^P f(x(\xi), \xi),$$

is σ_p-lsc on L^p.