Helsinki University of Technology
Mathematics

Somersalo/Bingham

Mat-1.198 Scattering Theory

5th set of exercises, 5.3.2003

Consider the scattering problem by a sound-hard obstacle: Given a solution $u_{\rm inc}$ of

$$(\triangle + k^2)u_{\rm inc} = 0 \qquad \text{in } \mathbb{R}^3,$$

find $u = u_{\rm inc} + u_{\rm sc}$ such that

$$\begin{aligned}(\triangle + k^2)u &= 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D} \\ \left.\frac{\partial u}{\partial n}\right|_{\partial D} &= 0 \\ \lim_{r\to\infty} r\left(\frac{\partial u_{\rm sc}}{\partial r} - iku_{\rm sc}\right) &= 0.\end{aligned}$$

1. Find an integral equation for $u|_{\partial D}$ using the Helmholtz representation theorem.

2. Formulate the result equivalent to Theorem 11.8 (corollary of the Fredholm Alternative) for this problem.

3. Consider the previous week's exercise concerning the scattering by a unit disc. By using Theorem 11.8 and the explicit form of the Neumann resonances in D, show that the equation
$$(1 + 2K')\varphi = 2\frac{\partial u_{\rm inc}}{\partial n}, \qquad u_{\rm inc}(x) = e^{ik\hat{\alpha}\cdot x}$$
is solvable even at resonance frequencies.

4. Prove **Theorem 11.2:**

   (1) If $A : X \to X$ is compact and $B : X \to X$ is bounded, then both AB and BA are compact.

   (2) If $A_n : X \to X$ is compact for all $n \in \mathbb{N}$ and $\|A_n - A\| \to 0$, then A is compact.

   (3) If $A : X \to X$ is bounded and finite dimensional, i.e., $\dim \operatorname{Ran}(A) < \infty$, *then A is compact.*